

DIRECT AND INVERSE SPECTRAL THEORY OF SINGULAR LEFT-DEFINITE STURM-LIOUVILLE OPERATORS

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ABSTRACT. We discuss direct and inverse spectral theory of self-adjoint Sturm–Liouville relations with separated boundary conditions in the left-definite setting. In particular, we develop singular Weyl–Titchmarsh theory for these relations. Consequently, we apply de Branges’ subspace ordering theorem to obtain inverse uniqueness results for the associated spectral measure. The results can be applied to solve the inverse spectral problem associated with the Camassa–Holm equation.

1. INTRODUCTION

Consider the left-definite Sturm–Liouville problem

$$(1.1) \quad -\frac{d}{dx} \left(p(x) \frac{d}{dx} y(x) \right) + q(x)y(x) = zr(x)y(x)$$

on some interval (a, b) . Here, by left-definite we mean that the real-valued function r is allowed to change sign but p and q are assumed to be non-negative. In the case of a regular left endpoint, Bennewitz [5], Brown and Weikard [7] recently developed Weyl–Titchmarsh theory for such problems, analogously to the right-definite case. Moreover, they were also able to prove that the associated spectral measure uniquely determines the left-definite Sturm–Liouville problem up to a so-called Liouville transform.

In the present paper we give an alternative proof of this result, using de Branges’ subspace ordering theorem for Hilbert spaces of entire functions. In fact, this approach allows us to deal with a larger class of problems. For instance, we allow the left endpoint to be quite singular and the weight function r to be a real-valued Borel measure. However, at a second glance our approach is not too different from the approach taken in [5] and [7]. The authors there prove Paley–Wiener type results to describe the spectral transforms of functions with compact support in order to obtain an appropriate Liouville transform. We will show that these spaces of transforms are actually hyperplanes in some de Branges spaces associated with our left-definite Sturm–Liouville problem. This will allow us to apply de Branges’ subspace ordering theorem to obtain a suitable Liouville transform.

As in [5] and [7], the main motivation for this work is the Camassa–Holm equation, an integrable, non-linear wave equation which models unidirectional propagation of waves on shallow water. Due to its many remarkable properties, this equation has gotten a lot of attention recently and we only refer to e.g. [8], [9], [10],

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[11], [23] for further information. Associated with the Camassa–Holm equation is the left-definite Sturm–Liouville problem

$$(1.2) \quad -y''(x) + \frac{1}{4}y(x) = z\omega(x)y(x)$$

on the real line. Direct, and in particular inverse spectral theory of this weighted Sturm–Liouville problem are of peculiar interest for solving the Camassa–Holm equation. Provided ω is strictly positive and smooth enough, it is possible to transform this problem into a Sturm–Liouville problem in potential form and some inverse spectral theory may be drawn from this. However, in order to incorporate the main interesting phenomena (wave breaking [10] and multi-peakon solutions [2], [12]) of the dispersionless Camassa–Holm equation, it is necessary to allow ω at least to be an arbitrary finite signed Borel measure on \mathbb{R} . In [5], [7] the authors were able to prove an inverse uniqueness result under some restrictions on the measure ω , which for example prohibits the case of multi-peakon solutions of the Camassa–Holm equation. Using the results of the present paper we are able to avoid these restrictions and to cover the case of arbitrary real finite measures ω ; see [17].

Note that this application also requires us to consider our Sturm–Liouville problem (1.1) with measure coefficients. For further information on measure Sturm–Liouville equations see e.g. [4] or [18] and the references therein. Moreover, the fact that we allow the weight measure to vanish on arbitrary sets, makes it necessary to work with linear relations instead of operators. Regarding the notion of linear relations, we refer to e.g. [1], [13], [15], [16], [21] or for a brief review, containing all facts which are needed here [18, Appendix B].

The paper is organized as follows. After some preliminaries about left-definite measure Sturm–Liouville equations, we give a review of linear relations associated with (1.1) in a modified Sobolev space. In particular, we discuss self-adjoint realizations with separated boundary conditions in Section 3. Since a lot of this first part are minor generalizations of results in e.g. [6], [7], [18], we will omit most of the proofs. In the consecutive two sections we develop Weyl–Titchmarsh theory for such self-adjoint relations. This part can essentially be done along the lines of the singular Weyl–Titchmarsh theory, recently introduced in [20] and [25] for Schrödinger operators. Section 6 introduces some de Branges spaces associated with a left-definite self-adjoint Sturm–Liouville problem. Moreover, we provide crucial properties of these spaces, which are needed for the proof of our inverse uniqueness result, which is carried out in the last section. In particular, this last section provides an inverse uniqueness result, which applies to the isospectral problem of the Camassa–Holm equation. Finally, in the appendix we give a brief review of de Branges’ theory of Hilbert spaces of entire functions as far as it is needed.

Before we start, let us recall some facts about functions which are absolutely continuous with respect to some measure. Therefore let (a, b) be an arbitrary interval and μ be a complex-valued Borel measure on (a, b) . With $AC_{\text{loc}}((a, b); \mu)$ we denote the set of all left-continuous functions, which are locally absolutely continuous with respect to the measure μ . These are precisely the functions f which can be written in the form

$$f(x) = f(c) + \int_c^x h(s) d\mu(s), \quad x \in (a, b)$$

for some $h \in L^1_{\text{loc}}((a, b); \mu)$, where the integral has to be read as

$$\int_c^x h(s) d\mu(s) = \begin{cases} \int_{[c, x)} h(s) d\mu(s), & \text{if } x > c, \\ 0, & \text{if } x = c, \\ -\int_{[x, c)} h(s) d\mu(s), & \text{if } x < c. \end{cases}$$

The function h is the Radon–Nikodým derivative of f with respect to μ . It is uniquely defined in $L^1_{\text{loc}}((a, b); \mu)$ and we write $df/d\mu = h$. Every function f which is locally absolutely continuous with respect to μ is locally of bounded variation and hence also its right-hand limits

$$f(x+) = \lim_{\varepsilon \downarrow 0} f(x + \varepsilon), \quad x \in (a, b)$$

exist everywhere and may differ from $f(x)$ only if μ has mass in x . Furthermore, we will repeatedly use the following integration by parts formula for complex-valued Borel measures μ, ν on (a, b) (see e.g. [22, Theorem 21.67])

$$(1.3) \quad \int_{\alpha}^{\beta} F(x) d\nu(x) = FG|_{\alpha}^{\beta} - \int_{\alpha}^{\beta} G(x+) d\mu(x), \quad \alpha, \beta \in (a, b),$$

where F, G are left-continuous distribution functions of μ, ν respectively.

2. LEFT-DEFINITE STURM–LIOUVILLE EQUATIONS

Let (a, b) be an arbitrary interval and ϱ, ς and χ be complex-valued Borel measures on (a, b) . We are going to define a linear differential expression τ which is informally given by

$$\tau f = \frac{d}{d\varrho} \left(-\frac{df}{d\varsigma} + \int f d\chi \right).$$

In the rest of this paper we will always assume that our measures satisfy the following four properties.

Hypothesis 2.1.

- (i) *The measure ϱ is real-valued.*
- (ii) *The measure ς is positive and supported on the whole interval.*
- (iii) *The measure χ is positive but not identically zero.*
- (iv) *The measure ς has no point masses in common with χ or ϱ , i.e.*

$$\varsigma(\{x\})\chi(\{x\}) = \varsigma(\{x\})\varrho(\{x\}) = 0, \quad x \in (a, b).$$

The maximal domain \mathfrak{D}_{τ} of functions such that the expression τf makes sense consists of all functions $f \in AC_{\text{loc}}((a, b); \varsigma)$ for which the function

$$(2.1) \quad -\frac{df}{d\varsigma}(x) + \int_c^x f d\chi, \quad x \in (a, b)$$

is locally absolutely continuous with respect to ϱ , i.e. there is some representative of this function lying in $AC_{\text{loc}}((a, b); \varrho)$. As a consequence of the assumption on the support of ς , this representative is unique. We then set $\tau f \in L^1_{\text{loc}}((a, b); \varrho)$ to be the Radon–Nikodým derivative of this function with respect to ϱ . One easily sees that this definition is independent of $c \in (a, b)$ since the corresponding functions (2.1) as well as their unique representatives only differ by an additive constant. As usual, the Radon–Nikodým derivative with respect to ς of some $f \in \mathfrak{D}_{\tau}$ is denoted with $f^{[1]}$ and referred to as the quasi-derivative of f .

Note that this definition includes classical Sturm–Liouville and Jacobi expressions as special cases. The following existence and uniqueness theorem for solutions of measure Sturm–Liouville equations may be found in [18, Theorem 3.1].

Theorem 2.2. *For each $g \in L^1_{\text{loc}}((a, b); \varrho)$, $c \in (a, b)$, $d_1, d_2 \in \mathbb{C}$ and $z \in \mathbb{C}$ there is a unique solution of the initial value problem*

$$(\tau - z)f = g \quad \text{with} \quad f(c) = d_1 \quad \text{and} \quad f^{[1]}(c) = d_2.$$

We say that τ is regular at an endpoint if the measures ϱ , ς and χ are finite near this endpoint. In this case, if g is also integrable near this endpoint, then each solution of the equation $(\tau - z)f = g$ may be continuously extended to this endpoint. Moreover, the initial point c in Theorem 2.2 may be chosen as this endpoint (see e.g. [18, Theorem 3.5]).

Associated with our differential expression τ is a linear relation T_{loc} in the space $AC_{\text{loc}}((a, b); \varsigma)$ defined by

$$T_{\text{loc}} = \{(f, f_\tau) \in AC_{\text{loc}}((a, b); \varsigma)^2 \mid f \in \mathfrak{D}_\tau, \tau f = f_\tau \text{ in } L^1_{\text{loc}}((a, b); \varrho)\}.$$

Regarding notation we will make the following convention. Given some pair $f \in T_{\text{loc}}$ we will denote its first component also with f and the second component with f_τ . Moreover, if $g \in AC_{\text{loc}}((a, b); \varsigma)$ and f is a solution of $(\tau - z)f = g$ for some $z \in \mathbb{C}$, then this solution f will often be identified with the pair $(f, g + zf) \in T_{\text{loc}}$.

In the right-definite theory, a crucial role is played by the Wronskian of two functions and the associated Lagrange identity. The corresponding quantity in the left-definite case is the function

$$(2.2) \quad V(f, g^*)(x) = f_\tau(x)g^{[1]}(x)^* - f^{[1]}(x)g_\tau(x)^*, \quad x \in (a, b),$$

defined for all pairs $f, g \in T_{\text{loc}}$. Using the integration by parts formula (1.3) and property (iv) in Hypothesis 2.1 one obtains the following Lagrange identity for this modified Wronskian.

Proposition 2.3. *For every $f, g \in T_{\text{loc}}$ and $\alpha, \beta \in (a, b)$ we have*

$$\begin{aligned} V(f, g^*)(\beta) - V(f, g^*)(\alpha) &= \int_\alpha^\beta f_\tau(x)g(x)^* - f(x)g_\tau(x)^* d\chi(x) \\ &\quad + \int_\alpha^\beta f^{[1]}_\tau(x)g^{[1]}(x)^* - f^{[1]}(x)g^{[1]}_\tau(x)^* d\varsigma(x). \end{aligned}$$

As a consequence of this Lagrange identity one sees that for each $z \in \mathbb{C}$ the modified Wronskian $V(u_1, u_2)$ of two solutions u_1, u_2 of $(\tau - z)u = 0$ is constant. Furthermore, for $z \neq 0$ the solutions u_1, u_2 are linearly dependent if and only if $V(u_1, u_2) = 0$. Another useful identity for the modified Wronskian is the following Plücker identity, which follows similarly as in [18, Proposition 3.4].

Proposition 2.4. *For every $f_1, f_2, f_3, f_4 \in T_{\text{loc}}$ we have*

$$0 = V(f_1, f_2)V(f_3, f_4) + V(f_1, f_3)V(f_4, f_2) + V(f_1, f_4)V(f_2, f_3).$$

In order to obtain a linear relation in a Hilbert space, we introduce a modified Sobolev space $H^1(a, b)$. It consists of all functions f on (a, b) which are locally absolutely continuous with respect to ς such that f is square integrable with respect

to χ and the Radon–Nikodým derivative $df/d\varsigma$ is square integrable with respect to ς . The space $H^1(a, b)$ is equipped with the inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x)^* d\chi(x) + \int_a^b f^{[1]}(x)g^{[1]}(x)^* d\varsigma(x), \quad f, g \in H^1(a, b).$$

Hereby note that f and g are always continuous in points of mass of χ in virtue of property (iv) in Hypothesis 2.1. It is not surprising that this modified Sobolev space turns out to be a reproducing kernel Hilbert space (see e.g. [7, Section 2]).

In order to obtain the maximal relation T_{\max} in $H^1(a, b)$ associated with our differential expression τ we restrict T_{loc} by

$$T_{\max} = \{(f, f_\tau) \in H^1(a, b) \times H^1(a, b) \mid (f, f_\tau) \in T_{\text{loc}}\}.$$

The following characterization of T_{\max} as weak solutions of our differential equation will be quite useful (the proof can be done along the lines of [7, Proposition 2.4]).

Proposition 2.5. *Some $(f, f_\tau) \in H^1(a, b) \times H^1(a, b)$ lies in T_{\max} if and only if*

$$(2.3) \quad \int_a^b f(x)g(x)^* d\chi(x) + \int_a^b f^{[1]}(x)g^{[1]}(x)^* d\varsigma(x) = \int_a^b f_\tau(x)g(x)^* d\varrho(x)$$

for each $g \in H_c^1(a, b)$.

Here, $H_c^1(a, b)$ denotes the linear subspace of $H^1(a, b)$ consisting of all functions with compact support. Consequently, some function $h \in H^1(a, b)$ lies in the multi-valued part of T_{\max} if and only if $h = 0$ almost everywhere with respect to $|\varrho|$.

We say some function $f \in AC_{\text{loc}}((a, b); \varsigma)$ lies in $H^1(a, b)$ near an endpoint if f is square integrable with respect to χ near this endpoint and its quasi-derivative is square integrable with respect to ς near this endpoint. Furthermore, we say some pair $f \in T_{\text{loc}}$ lies in T_{\max} near an endpoint if both components f and f_τ lie in $H^1(a, b)$ near this endpoint. Clearly, some $f \in T_{\text{loc}}$ lies in T_{\max} if and only if it lies in T_{\max} near a and near b . Using the Lagrange identity one shows the following properties of the modified Wronskian on T_{\max} .

Lemma 2.6. *If f and g lie in T_{\max} near an endpoint, then the limit of $V(f, g^*)(x)$ as x tends to this endpoint exists and is finite. If f and g even lie in T_{\max} , then*

$$(2.4) \quad \langle f_\tau, g \rangle - \langle f, g_\tau \rangle = V(f, g^*)(b) - V(f, g^*)(a).$$

Moreover, $V(\cdot, \cdot)(a)$ and $V(\cdot, \cdot)(b)$ are continuous bilinear forms on T_{\max} with respect to the product topology on T_{\max} .

If τ is regular at an endpoint, then it is not hard to see that for each f which lies in T_{\max} near this endpoint, the limits of $f(x)$, $f^{[1]}(x)$ and $f_\tau(x)$ as x tends to this endpoint exist and are finite. Of course in this case, equation (2.2) extends to this regular endpoint provided that f and g lie in T_{\max} near this endpoint.

Next we will collect some more properties of the modified Sobolev space $H^1(a, b)$ and the maximal relation T_{\max} . The next proposition may be proved similarly to [7, Theorem 2.6] and [7, Proposition 2.7]. Here and in the following, $H_0^1(a, b)$ will denote the closure of $H_c^1(a, b)$ in $H^1(a, b)$.

Proposition 2.7. *We have $H^1(a, b) = H_0^1(a, b) \oplus \ker(T_{\max})$, with*

$$(2.5) \quad \dim \ker(T_{\max}) = \begin{cases} 0, & \text{if } \varsigma + \chi \text{ is infinite near both endpoints,} \\ 1, & \text{if } \varsigma + \chi \text{ is finite near precisely one endpoint,} \\ 2, & \text{if } \varsigma + \chi \text{ is finite.} \end{cases}$$

Moreover, there are (up to scalar multiples) unique non-trivial real solutions w_a , w_b of $\tau u = 0$ which lie in $H^1(a, b)$ near a , b respectively and satisfy

$$(2.6) \quad \lim_{\alpha \rightarrow a} g(\alpha) w_a^{[1]}(\alpha) = \lim_{\beta \rightarrow b} g(\beta) w_b^{[1]}(\beta) = 0, \quad g \in H^1(a, b).$$

The solutions w_a and w_b are linearly independent.

Also note that the functions

$$(2.7) \quad x \mapsto w_a(x) w_a^{[1]}(x) \quad \text{and} \quad x \mapsto w_b(x) w_b^{[1]}(x)$$

are increasing on (a, b) and strictly positive and negative, respectively. Now for each fixed $c \in (a, b)$ we introduce the function

$$(2.8) \quad \delta_c(x) = \frac{1}{W(w_b, w_a)} \begin{cases} w_a(x) w_b(c), & \text{if } x \in (a, c], \\ w_a(c) w_b(x), & \text{if } x \in (c, b), \end{cases}$$

with the usual Wronskian of w_a and w_b

$$W(w_b, w_a) = w_b(x) w_a^{[1]}(x) - w_b^{[1]}(x) w_a(x),$$

where the right-hand side is independent of $x \in (a, b)$ and non-zero since w_a and w_b are linearly independent solutions of $\tau u = 0$. With this definition the point evaluation in c is given by

$$f(c) = \langle f, \delta_c \rangle, \quad f \in H^1(a, b).$$

More precisely, this follows from splitting the integrals on the right-hand side, integrating by parts twice and using the properties of the functions w_a , w_b from Proposition 2.7. Furthermore, if the measures ς and χ are finite near an endpoint, say a , then $f(x)$ has a finite limit as $x \rightarrow a$ for each $f \in H^1(a, b)$ and

$$f(a) = \lim_{\alpha \rightarrow a} f(\alpha) = \langle f, \delta_a \rangle, \quad f \in H^1(a, b),$$

where the function δ_a is given by

$$(2.9) \quad \delta_a(x) = -\frac{w_b(x)}{w_b^{[1]}(a)}, \quad x \in (a, b).$$

In fact, this follows from a simple integration by parts and Proposition 2.7. If ς and χ are finite near the right endpoint b , then obviously a similar result holds for b . As a consequence of this, some function $f \in H^1(a, b)$ lies in $H_0^1(a, b)$ if and only if f vanishes in each endpoint near which ς and χ are finite.

3. SELF-ADJOINT STURM-LIOUVILLE RELATIONS

In the present section we are interested in self-adjoint restrictions of the maximal relation T_{\max} . Therefore, we will first compute its adjoint relation.

Theorem 3.1. *The maximal relation T_{\max} is closed with adjoint given by*

$$(3.1) \quad T_{\max}^* = \{f \in T_{\max} \mid \forall g \in T_{\max} : V(f, g)(a) = V(f, g)(b) = 0\}.$$

Proof. Let $T_0 \subseteq T_{\max}$ consist of all $f \in T_{\max}$ such that $f_\tau \in H_c^1(a, b)$, f is a scalar multiple of w_a near a and a scalar multiple of w_b near b . Then the range of T_0 is actually equal to $H_c^1(a, b)$. Indeed, if $g \in H_c^1(a, b)$ is given, then the function

$$f(x) = W(w_b, w_a)^{-1} \left(w_b(x) \int_a^x w_a g \, d\varrho + w_a(x) \int_x^b w_b g \, d\varrho \right), \quad x \in (a, b)$$

is a solution of $\tau f = g$ (see [18, Proposition 3.3]) which is a scalar multiple of w_a, w_b near the respective endpoints and hence $g \in \text{ran}(T_0)$. Moreover, for each $f \in T_0, g \in T_{\max}$ the limits of $V(f, g)(x)$ as $x \rightarrow a$ and as $x \rightarrow b$ vanish in view of Proposition 2.7. Hence Lemma 2.6 shows that $T_{\max} \subseteq T_0^*$. Conversely, if $(f_1, f_2) \in T_0^*$, then integration by parts and Proposition 2.7 show that

$$\langle f_1, g_\tau \rangle = \langle f_2, g \rangle = \int_a^b f_2(x) g_\tau(x)^* d\varrho(x), \quad g \in T_0.$$

Now since $\text{ran}(T_0) = H_c^1(a, b)$ we infer that $(f_1, f_2) \in T_{\max}$ in view of Proposition 2.5. Thus T_{\max} is the adjoint of T_0 and hence closed. Finally we obtain

$$T_{\max}^* = \overline{T_0} \subseteq \{f \in T_{\max} \mid \forall g \in T_{\max} : V(f, g)(a) = V(f, g)(b) = 0\} \subseteq T_{\max}^*,$$

where we used Lemma 2.6 and the continuity of $V(\cdot, \cdot)(a)$ and $V(\cdot, \cdot)(b)$. \square

The adjoint of T_{\max} is referred to as the minimal relation T_{\min} . This linear relation is obviously symmetric with adjoint T_{\max} . Since T_{\min} is real with respect to the natural conjugation on $H^1(a, b)$, its deficiency indices are equal (see [18, Theorem 4.9]) and at most two because there are only two linearly independent solutions of $(\tau - i)u = 0$. In particular, this shows that T_{\max} always has self-adjoint restrictions. However, the actual deficiency index of T_{\min} depends on which cases in the following alternative (see [6, Lemma 4]) prevail. At each endpoint, either

- (i) for every $z \in \mathbb{C}^\times$ all solutions of $(\tau - z)u = 0$ lie in $H^1(a, b)$ near this endpoint or
- (ii) for every $z \in \mathbb{C}^\times$ there is a solution of $(\tau - z)u = 0$ which does not lie in $H^1(a, b)$ near this endpoint.

Here and henceforth, the cross indicates that zero is removed from the respective set. The former case (i) is referred to as the limit-circle (l.c.) case and the latter (ii) as the limit-point (l.p.) case. Unlike in the right-definite theory, there is a precise criterion for the l.c. case to prevail in terms of our measure coefficients. In fact, [6, Theorem 3] shows that τ is in the l.c. case at an endpoint if and only if ς, χ are finite near this endpoint and the function $\int_c^x d\varrho, x \in (a, b)$ is square integrable with respect to ς near this endpoint for some $c \in (a, b)$. Furthermore, this theorem also ensures that all solutions of $\tau u = 0$ lie in $H^1(a, b)$ near an endpoint, if τ is in the l.c. case there. However, note that it is possible that τ is in the l.p. case at an endpoint although all solutions of $\tau u = 0$ lie in $H^1(a, b)$ near this endpoint. Now along the lines of the corresponding proofs in the right-definite case [18, Section 5], one may show the following result.

Theorem 3.2. *The deficiency index of T_{\min} is given by*

$$(3.2) \quad n(T_{\min}) = \begin{cases} 0, & \text{if } \tau \text{ is in the l.c. case at no endpoint,} \\ 1, & \text{if } \tau \text{ is in the l.c. case at precisely one endpoint,} \\ 2, & \text{if } \tau \text{ is in the l.c. case at both endpoints.} \end{cases}$$

Furthermore, it is also possible to adapt the proof of [18, Lemma 5.6], which shows that one is able to tell from the modified Wronskian whether τ is in the l.c. or in the l.p. case.

Proposition 3.3. *The endpoint a is in the l.p. case if and only if $V(f, g)(a) = 0$ for every $f, g \in T_{\max}$. If a is in the l.c. case, then there is a $v \in T_{\max}$ with*

$$(3.3) \quad V(v, v^*)(a) = 0 \quad \text{and} \quad V(f, v^*)(a) \neq 0 \quad \text{for some } f \in T_{\max}.$$

Similar results hold at the endpoint b .

Because of the formal similarity with the right-definite theory, it is now easy to obtain a precise characterization of all self-adjoint restrictions of T_{\max} in terms of boundary conditions at all endpoints which are in the l.c. case. This can be done following literally the proofs in [18, Section 6]. However, since we are only interested in separated boundary conditions we only state the following result.

Theorem 3.4. *Let $v_a, v_b \in T_{\max}$ such that*

$$(3.4a) \quad V(v_a, v_a^*)(a) = 0 \quad \text{and} \quad V(f, v_a^*)(a) \neq 0 \quad \text{for some } f \in T_{\max},$$

$$(3.4b) \quad V(v_b, v_b^*)(b) = 0 \quad \text{and} \quad V(f, v_b^*)(b) \neq 0 \quad \text{for some } f \in T_{\max},$$

if τ is in the l.c. case at a, b , respectively. Then the linear relation

$$(3.5) \quad S = \{f \in T_{\max} \mid V(f, v_a^*)(a) = V(f, v_b^*)(b) = 0\}$$

is a self-adjoint restriction of T_{\max} .

Note that boundary conditions at endpoints which are in the l.p. case are superfluous, since in this case each $f \in T_{\max}$ satisfies them in view of Proposition 3.3. Furthermore, Theorem 3.4 actually gives all possible self-adjoint restrictions of T_{\max} provided that τ is not at both endpoints in the l.c. case.

If τ is regular at an endpoint, say a , then the boundary condition at this endpoint may be given in a simpler form. In fact, if some $v_a \in T_{\max}$ with (3.4a) is given, then it can be shown that there is some $\varphi_a \in [0, \pi)$ such that for each $f \in T_{\max}$

$$(3.6) \quad V(f, v_a^*)(a) = 0 \quad \Leftrightarrow \quad f_\tau(a) \cos \varphi_a - f^{[1]}(a) \sin \varphi_a = 0.$$

Conversely, if some $\varphi_a \in [0, \pi)$ is given, then there is a $v_a \in T_{\max}$ with (3.4a) such that (3.6) holds for all $f \in T_{\max}$. The boundary conditions corresponding to $\varphi_a = 0$ are called Dirichlet boundary conditions, whereas the ones corresponding to $\varphi_a = \pi/2$ are called Neumann boundary conditions. Moreover, note that for a solution of $(\tau - z)u = 0$ with $z \in \mathbb{C}$, the boundary condition at a takes the form

$$zu(a) \cos \varphi_a - u^{[1]}(a) \sin \varphi_a = 0.$$

As in [18, Corollary 8.4], one may show using Proposition 3.3 and the Plücker identity that all non-zero eigenvalues of self-adjoint restrictions S with separated boundary conditions are simple. However, it might happen that zero is a double eigenvalue indeed. This is due to the fact that there are cases in which all solutions of $\tau u = 0$ lie in T_{\max} and satisfy the possible boundary condition near some endpoint. For example, this happens for Dirichlet boundary conditions at a regular endpoint or if ς and χ are finite near an endpoint which is in the l.p. case.

Theorem 3.5. *Suppose S is a self-adjoint restriction of T_{\max} with separated boundary conditions and let $z \in \rho(S)^\times$. Furthermore, let u_a and u_b be non-trivial solutions of $(\tau - z)u = 0$ such that*

$$u_{a/b} \begin{cases} \text{satisfies the boundary condition at } a/b \text{ if } \tau \text{ is in the l.c. case at } a/b, \\ \text{lies in } H^1(a, b) \text{ near } a/b \text{ if } \tau \text{ is in the l.p. case at } a/b. \end{cases}$$

Then the resolvent R_z is given by

$$(3.7) \quad R_z g(x) = \langle g, G_z(x, \cdot)^* \rangle, \quad x \in (a, b), \quad g \in H^1(a, b),$$

where

$$(3.8) \quad G_z(x, y) + \frac{\delta_x(y)}{z} = \frac{1}{V(u_b, u_a)} \begin{cases} u_a(y)u_b(x), & \text{if } y \leq x, \\ u_a(x)u_b(y), & \text{if } y > x. \end{cases}$$

Proof. First of all, the solutions u_a, u_b are linearly independent, since otherwise z would be an eigenvalue of S . Now if $g \in H_c^1(a, b)$, then f_g given by

$$f_g(x) = \frac{z}{V(u_b, u_a)} \left(u_b(x) \int_a^x u_a g \, d\rho + u_a(x) \int_x^b u_b g \, d\rho \right), \quad x \in (a, b)$$

is a solution of $(\tau - z)f = g$ because of [18, Proposition 3.3]. Moreover, f_g is a scalar multiple of u_a near a and a scalar multiple of u_b near b . As a consequence $f_g \in T_{\max}$ satisfies the boundary conditions of S and therefore $R_z g = f_g$. Now an integration by parts shows that $R_z g$ is given as in (3.7). Furthermore, by continuity this holds for all $g \in H_0^1(a, b)$. Hence it remains to consider $R_z w$ for $w \in \ker(T_{\max})$. In this case integration by parts yields

$$\langle w, G_z(x, \cdot)^* \rangle = \frac{1}{z} \frac{V(u_b, w)(b)}{V(u_b, u_a)(b)} u_a(x) + \frac{1}{z} \frac{V(w, u_a)(a)}{V(u_b, u_a)(a)} u_b(x) - \frac{w(x)}{z}, \quad x \in (a, b).$$

Obviously, this function is a solution of $(\tau - z)f = w$, since w is a solution of $\tau u = 0$. Moreover, if τ is in the l.p. case at a , then the second term vanishes in view of Proposition 3.3. For the same reason the first term vanishes if τ is in the l.p. case at b and hence this function even lies in $H^1(a, b)$. Using the Plücker identity one sees that this function also satisfies all possible boundary conditions. \square

4. SINGULAR WEYL–TITCHMARSH FUNCTION

Let S be some self-adjoint restriction of T_{\max} with separated boundary conditions as in Theorem 3.4. In this section we will introduce a singular Weyl–Titchmarsh function as it has been done recently in [20], [25] and [18] for the right-definite case. To this end we first need a non-trivial real analytic solution ϕ_z , $z \in \mathbb{C}^\times$ of $(\tau - z)u = 0$ such that ϕ_z lies in S near a , i.e. ϕ_z lies in $H^1(a, b)$ near a and satisfies the boundary condition at a if τ is in the l.c. case there.

Hypothesis 4.1. *For each $z \in \mathbb{C}^\times$ there is a non-trivial solution ϕ_z of $(\tau - z)u = 0$ such that ϕ_z lies in S near a and the functions*

$$(4.1) \quad z \mapsto \phi_z(c) \quad \text{and} \quad z \mapsto \phi_z^{[1]}(c)$$

are real analytic in \mathbb{C}^\times with at most poles at zero for each $c \in (a, b)$.

In order to introduce a singular Weyl–Titchmarsh function we furthermore need a second real analytic solution θ_z , $z \in \mathbb{C}^\times$ of $(\tau - z)u = 0$ with $V(\theta_z, \phi_z) = 1$.

Lemma 4.2. *If Hypothesis 4.1 holds, then for each $z \in \mathbb{C}^\times$ there is a solution θ_z of $(\tau - z)u = 0$ such that $V(\theta_z, \phi_z) = 1$ and the functions*

$$(4.2) \quad z \mapsto \theta_z(c) \quad \text{and} \quad z \mapsto \theta_z^{[1]}(c)$$

are real analytic in \mathbb{C}^\times with at most poles at zero for each $c \in (a, b)$.

Proof. Following literally the proof of [25, Lemma 2.4] there is a real analytic solution u_z , $z \in \mathbb{C}^\times$ of $(\tau - z)u = 0$ such that the usual Wronskian satisfies

$$W(u_z, \phi_z) = u_z(x)\phi_z^{[1]}(x) - u_z^{[1]}(x)\phi_z(x) = 1, \quad x \in (a, b), \quad z \in \mathbb{C}^\times.$$

Now the solutions $\theta_z = z^{-1}u_z$, $z \in \mathbb{C}^\times$ have the claimed properties. \square

Given a real analytic fundamental system θ_z, ϕ_z , $z \in \mathbb{C}^\times$ of $(\tau - z)u = 0$ as in Hypothesis 4.1 and Lemma 4.2 we may define a complex valued function M on $\rho(S)^\times$ by requiring that the solutions

$$(4.3) \quad \psi_z = \theta_z + M(z)\phi_z, \quad z \in \rho(S)^\times$$

lie in S near b , i.e. they lie in T_{\max} near b and satisfy the boundary condition at b if τ is in the l.c. case there. Because of Theorem 3.2 and the fact that there is up to scalar multiples precisely one solution of $(\tau - z)u = 0$ satisfying the boundary condition at b if τ is in the l.c. case there, M is well-defined and referred to as the singular Weyl–Titchmarsh function of S , associated with the fundamental system θ_z, ϕ_z , $z \in \mathbb{C}^\times$.

Theorem 4.3. *The singular Weyl–Titchmarsh function M is analytic with*

$$(4.4) \quad M(z) = M(z^*)^*, \quad z \in \rho(S)^\times.$$

Proof. From Theorem 3.5 we get for each $c \in (a, b)$

$$(4.5) \quad \begin{aligned} \langle R_z \delta_c, \delta_c \rangle &= R_z \delta_c(c) = G_z(c, c) = \psi_z(c)\phi_z(c) - \frac{\delta_c(c)}{z} \\ &= M(z)\phi_z(c)^2 + \theta_z(c)\phi_z(c) - \frac{w_a(c)w_b(c)}{zW(w_b, w_a)}, \quad z \in \rho(S)^\times, \end{aligned}$$

which proves the claim. \square

Similarly to the right-definite case (see e.g. [25, Lemma A.4], [18, Theorem 9.4]), it is possible to construct a real analytic fundamental system θ_z, ϕ_z , $z \in \mathbb{C}^\times$ of $(\tau - z)u = 0$ as in Hypothesis 4.1 and Lemma 4.2, if τ is in the l.c. case at a .

Proposition 4.4. *Suppose τ is in the l.c. case at a . Then there is a real analytic fundamental system θ_z, ϕ_z , $z \in \mathbb{C}^\times$ of $(\tau - z)u = 0$ as in Hypothesis 4.1 and Lemma 4.2 such that for all $z_1, z_2 \in \mathbb{C}^\times$ we additionally have*

$$(4.6) \quad V(\theta_{z_1}, \phi_{z_2})(a) = 1 \quad \text{and} \quad V(\theta_{z_1}, \theta_{z_2})(a) = V(\phi_{z_1}, \phi_{z_2})(a) = 0.$$

In this case, the corresponding function M is a Herglotz–Nevanlinna function.

In particular, if τ is regular at a and the boundary condition there is given by

$$f_\tau(a) \cos \varphi_a - f^{[1]}(a) \sin \varphi_a = 0, \quad f \in S$$

for some $\varphi_a \in [0, \pi)$, then a real analytic fundamental system θ_z, ϕ_z , $z \in \mathbb{C}^\times$ of $(\tau - z)u = 0$ is given for example by the initial conditions

$$z\phi_z(a) = -\theta_z^{[1]}(a) = \sin \varphi_a \quad \text{and} \quad \phi_z^{[1]}(a) = z\theta_z(a) = \cos \varphi_a, \quad z \in \mathbb{C}^\times.$$

Obviously, this fundamental system satisfies the properties in Proposition 4.4.

As in the right-definite case (see [20, Lemma 3.2], [25, Lemma 2.2], [18, Theorem 9.6]), we may give a necessary and sufficient condition for Hypothesis 4.1 to hold. Therefore fix some $c \in (a, b)$ such that $\chi((a, c)) \neq 0$ and consider the maximal relation in $H^1(a, c)$ associated with our differential expression restricted to

(a, c) . With S_c we denote the self-adjoint restriction of this relation with the same boundary conditions as S near a and Dirichlet boundary conditions at c .

Lemma 4.5. *Hypothesis 4.1 holds if and only if the self-adjoint relation S_c has purely discrete spectrum.*

This lemma can be proved along the lines of [25, Lemma 2.2] and [18, Theorem 9.6]. Moreover, if Hypothesis 4.1 holds at both endpoints, then it turns out that the spectrum of S is purely discrete. In particular, S has purely discrete spectrum provided that τ is in the l.c. case at both endpoints.

5. SPECTRAL TRANSFORMATION

In this section let S again be a self-adjoint restriction of T_{\max} with separated boundary conditions such that Hypothesis 4.1 holds. For the sake of simplicity we will furthermore assume that zero is not an eigenvalue of S . This excludes for example the case of Dirichlet boundary conditions at a regular endpoint or if ς and χ are finite near an endpoint which is in the l.p. case.

Next recall that for all functions $f, g \in H^1(a, b)$ there is a unique complex Borel measure $E_{f,g}$ on \mathbb{R} such that

$$(5.1) \quad \langle R_z f, g \rangle = \int_{\mathbb{R}} \frac{1}{\lambda - z} dE_{f,g}(\lambda), \quad z \in \rho(S).$$

In fact, these measures are obtained by applying a variant of the spectral theorem to the operator part of S (see e.g. [18, Lemma B.4]).

Lemma 5.1. *There is a unique Borel measure μ on \mathbb{R}^\times such that*

$$(5.2) \quad E_{\delta_\alpha, \delta_\beta}(B) = \int_{B^\times} \phi_\lambda(\alpha) \phi_\lambda(\beta) d\mu(\lambda)$$

for all $\alpha, \beta \in (a, b)$ and each Borel set $B \subseteq \mathbb{R}$.

Proof. As in the proof of Theorem 4.3 one sees that for $\alpha, \beta \in (a, b)$

$$\langle R_z \delta_\alpha, \delta_\beta \rangle = M(z) \phi_z(\alpha) \phi_z(\beta) + H_{\alpha, \beta}(z), \quad z \in \rho(S)^\times,$$

where $H_{\alpha, \beta}$ is a real analytic function on \mathbb{C}^\times . Now the claim may be deduced following the arguments in the proof of [25, Lemma 3.3]. \square

Now similar to [25, Theorem 3.4] and [18, Lemma 10.2], we may introduce a spectral transformation for our self-adjoint linear relation S .

Lemma 5.2. *There is a unique bounded linear operator $\mathcal{F} : H^1(a, b) \rightarrow L^2(\mathbb{R}^\times; \mu)$ such that for each $c \in (a, b)$ we have $\mathcal{F} \delta_c(\lambda) = \phi_\lambda(c)$ for almost all $\lambda \in \mathbb{R}^\times$ with respect to μ . The operator \mathcal{F} is a surjective partial isometry with initial subspace $\text{dom}(S)$. Its adjoint is given by*

$$(5.3) \quad \mathcal{F}^* g(x) = \int_{\mathbb{R}^\times} \phi_\lambda(x) g(\lambda) d\mu(\lambda), \quad x \in (a, b), \quad g \in L^2(\mathbb{R}^\times; \mu),$$

its (in general multi-valued) inverse is given by

$$(5.4) \quad \mathcal{F}^{-1} = \{(g, f) \in L^2(\mathbb{R}^\times; \mu) \times H^1(a, b) \mid \mathcal{F}^* g - f \in \text{mul}(S)\}.$$

If τ is in the l.c. case at b , then the transform of a function $f \in H^1(a, b)$ is

$$\mathcal{F}f(\lambda) = \langle \phi_\lambda, f^* \rangle = \int_a^b \phi_\lambda(x) f(x) d\chi(x) + \int_a^b \phi_\lambda^{[1]}(x) f^{[1]}(x) d\zeta(x), \quad \lambda \in \mathbb{R}^\times$$

by continuity. Of course, if τ is in the l.p. case at b , then this is not possible since ϕ_λ does not lie in $H^1(a, b)$ unless λ is an eigenvalue. However, we still have the following general result.

Proposition 5.3. *If $f \in H^1(a, b)$ vanishes near b , then*

$$(5.5) \quad \mathcal{F}f(\lambda) = \int_a^b \phi_\lambda(x) f(x) d\chi(x) + \int_a^b \phi_\lambda^{[1]}(x) f^{[1]}(x) d\zeta(x)$$

for almost all $\lambda \in \mathbb{R}^\times$ with respect to μ .

Proof. First of all, an integration by parts shows that for $\lambda \in \mathbb{R}^\times$ and $c \in (a, b)$

$$(*) \quad \int_a^x \phi_\lambda \delta_c d\chi + \int_a^x \phi_\lambda^{[1]} \delta_c^{[1]} d\zeta = \phi_\lambda(c) + \frac{w_a(c)w_b^{[1]}(x)}{W(w_b, w_a)} \phi_\lambda(x), \quad x \in (c, b).$$

Now pick some $\beta \in (a, b)$ such that f vanishes on $[\beta, b)$ and consider the space H_β of functions in $H^1(a, b)$ which are equal to a scalar multiple of w_b on $[\beta, b)$. It is not hard to see that this space is closed and that it contains all functions δ_c , $c \leq \beta$. Moreover, the linear span of these functions is even dense in H_β , i.e. f lies in the closure of $\text{span}\{\delta_c \mid c \leq \beta\}$. Now for each $k \in \mathbb{N}$ let $N(k) \in \mathbb{N}$ and $a_n^k \in \mathbb{C}$, $c_n^k \in (a, \beta)$ for $n = 1, \dots, N(k)$ such that the functions

$$f_k(x) = \sum_{n=1}^{N(k)} a_n^k \delta_{c_n^k}(x), \quad x \in (a, b), \quad k \in \mathbb{N}$$

converge to f in $H^1(a, b)$ as $k \rightarrow \infty$. Using equation $(*)$ we may estimate

$$\begin{aligned} & \left| \int_a^b \phi_\lambda(x) f(x) d\chi(x) + \int_a^b \phi_\lambda^{[1]}(x) f^{[1]}(x) d\zeta(x) - \sum_{n=1}^{N(k)} a_n^k \phi_\lambda(c_n^k) \right| \\ & \leq \left| \int_a^\beta \phi_\lambda(f - f_k) d\chi + \int_a^\beta \phi_\lambda^{[1]}(f^{[1]} - f_k^{[1]}) d\zeta \right| + \left| \phi_\lambda(\beta) \frac{w_b^{[1]}(\beta)}{w_b(\beta)} f_k(\beta) \right| \end{aligned}$$

for each $\lambda \in \mathbb{R}^\times$. The first term converges to zero since f_k converges to f in $H^1(a, b)$ as $k \rightarrow \infty$. Moreover, the second term converges to zero since $f_k(\beta)$ converges to zero as $k \rightarrow \infty$. But this proves the claim since $\mathcal{F}f_k(\lambda)$ converges to $\mathcal{F}f(\lambda)$ for almost all $\lambda \in \mathbb{R}^\times$ with respect to μ . \square

If F is a Borel measurable function on \mathbb{R}^\times , then we denote with M_F the maximally defined operator of multiplication with F in $L^2(\mathbb{R}^\times; \mu)$. We are now ready to state the main theorem of this section, which may be proved similarly to [18, Theorem 10.3].

Theorem 5.4. *The self-adjoint relation S is given by $S = \mathcal{F}^{-1} M_{\text{id}} \mathcal{F}$.*

Note that all of the multi-valuedness of S is only contained in the inverse of our spectral transformation. Moreover, the self-adjoint operator part of S is unitarily equivalent to the operator of multiplication M_{id} in $L^2(\mathbb{R}^\times; \mu)$. In fact, \mathcal{F} is unitary as an operator from $\overline{\text{dom}(S)}$ onto $L^2(\mathbb{R}^\times; \mu)$ and maps the operator part of S onto

multiplication with the independent variable. Now the spectrum of S can be read off from the boundary behavior of the singular Weyl–Titchmarsh function M in the usual way (see [25, Corollary 3.5])

$$(5.6) \quad \sigma(S)^\times = \text{supp}(\mu) = \overline{\{\lambda \in \mathbb{R}^\times \mid 0 < \limsup_{\varepsilon \downarrow 0} \text{Im } M(\lambda + i\varepsilon)\}}.$$

Moreover, the point spectrum of S is given by

$$(5.7) \quad \sigma_p(S) = \{\lambda \in \mathbb{R}^\times \mid \lim_{\varepsilon \downarrow 0} \varepsilon \text{Im } M(\lambda + i\varepsilon) > 0\},$$

with $\mu(\{\lambda\}) = \|\phi_\lambda\|^{-2}$ for all eigenvalues $\lambda \in \sigma_p(S)$.

Finally, note that the measure μ is uniquely determined by the property that the mapping $\delta_c \mapsto \phi_\lambda(c)$, $c \in (a, b)$ uniquely extends to a partial isometry onto $L^2(\mathbb{R}^\times; \mu)$, which maps S onto multiplication with the independent variable. Because of this, the measure μ is referred to as the spectral measure of S associated with the real analytic solutions ϕ_z , $z \in \mathbb{C}^\times$.

6. ASSOCIATED DE BRANGES SPACES

As in the previous sections let S be some self-adjoint restriction of T_{\max} (with separated boundary conditions) which does not have zero as an eigenvalue. The aim of the present section is to describe the spaces of transforms of functions in $H^1(a, b)$ with compact support. It will turn out that these spaces are hyperplanes in some de Branges spaces associated with our left-definite Sturm–Liouville problem, at least if we somewhat strengthen Hypothesis 4.1. In fact, in this section we will assume that for each $z \in \mathbb{C}$ there is a non-trivial solution ϕ_z of $(\tau - z)u = 0$ such that ϕ_z lies in S near a and the functions

$$z \mapsto \phi_z(c) \quad \text{and} \quad z \mapsto \phi_z^{[1]}(c)$$

are real entire for each $c \in (a, b)$. In particular, note that the solution ϕ_0 is always a scalar multiple of the solution w_a (due to the assumption that zero is not an eigenvalue of S). For example, if τ is regular at a and the boundary condition at a is given by (3.6) for some $\varphi_a \in (0, \pi)$, then such a real entire solution ϕ_z , $z \in \mathbb{C}$ of $(\tau - z)u = 0$ is given by the initial conditions

$$\phi_z(a) = \sin \varphi_a \quad \text{and} \quad \phi_z^{[1]}(a) = z \cos \varphi_a, \quad z \in \mathbb{C}.$$

Furthermore, we will assume that the measure ς is absolutely continuous with respect to the Lebesgue measure. This will guarantee that our chain of de Branges spaces is continuous in some sense, which simplifies the discussion to some extent. However, we do not have to impose additional assumptions on the measures χ and ϱ .

First of all we will introduce the de Branges spaces associated with S and our real entire solution ϕ_z , $z \in \mathbb{C}$. For a brief review of de Branges' theory of Hilbert spaces of entire functions see Appendix A, whereas for a detailed account we refer to de Branges' book [14]. Now fix some $c \in (a, b)$ and consider the entire function

$$(6.1) \quad E(z, c) = z\phi_z(c) + i\phi_z^{[1]}(c), \quad z \in \mathbb{C}.$$

Then this function is a de Branges function, i.e. it satisfies

$$|E(z, c)| > |E(z^*, c)|, \quad z \in \mathbb{C}^+,$$

where \mathbb{C}^+ is the open upper complex half-plane. Indeed, a simple calculation, using the Lagrange identity from Proposition 2.3 shows that

$$\begin{aligned} \frac{E(z, c)E^\#(\zeta^*, c) - E(\zeta^*, c)E^\#(z, c)}{2i(\zeta^* - z)} &= \frac{\zeta^* \phi_\zeta(c)^* \phi_z^{[1]}(c) - z \phi_z(c) \phi_\zeta^{[1]}(c)^*}{\zeta^* - z} \\ &= \int_a^c \phi_\zeta^* \phi_z d\chi + \int_a^c \phi_\zeta^{[1]*} \phi_z^{[1]} d\varsigma \end{aligned}$$

for each $\zeta, z \in \mathbb{C}^+$. In particular, choosing $\zeta = z$ this equality shows that our function $E(\cdot, c)$ is a de Branges function. Hence it gives rise to a de Branges space $B(c)$ equipped with the inner product

$$[F, G]_{B(c)} = \frac{1}{\pi} \int_{\mathbb{R}} \frac{F(\lambda)G(\lambda)^*}{|E(\lambda, c)|^2} d\lambda, \quad F, G \in B(c).$$

Moreover, note that $E(\cdot, c)$ does not have any real zeros λ . Indeed, if $\lambda \neq 0$ this would mean that both, ϕ_λ and its quasi-derivative vanish in c and if $\lambda = 0$ this would contradict the fact that ϕ_0 is a scalar multiple of w_a .

The reproducing kernel $K(\cdot, \cdot, c)$ of the de Branges space $B(c)$ is given as in equation (A.1). A similar calculation as above, using the Lagrange identity shows that it may be written as

$$(6.2) \quad K(\zeta, z, c) = \int_a^c \phi_\zeta(x)^* \phi_z(x) d\chi(x) + \int_a^c \phi_\zeta^{[1]}(x)^* \phi_z^{[1]}(x) d\varsigma(x), \quad \zeta, z \in \mathbb{C}.$$

In the following, the function $K(0, \cdot, c)$ will be of particular interest. An integration by parts shows that this function may as well be written as

$$(6.3) \quad K(0, z, c) = \phi_0^{[1]}(c) \phi_z(c), \quad z \in \mathbb{C},$$

where the boundary term at a vanishes since ϕ_0 is a scalar multiple of w_a .

We want to link the de Branges space $B(c)$ to our generalized Fourier transform \mathcal{F} , using Proposition 5.3. Therefore consider the modified Sobolev space $H^1(a, c)$ and define the transform of a function $f \in H^1(a, c)$ as

$$(6.4) \quad \hat{f}(z) = \int_a^c \phi_z(x) f(x) d\chi(x) + \int_a^c \phi_z^{[1]}(x) f^{[1]}(x) d\varsigma(x), \quad z \in \mathbb{C}.$$

We will now identify the de Branges space $B(c)$ with the space of transforms of functions from the subspace

$$D(c) = \overline{\text{span}\{\phi_z|_{(a,c)} \mid z \in \mathbb{C}\}}$$

of $H^1(a, c)$, equipped with the norm inherited from $H^1(a, c)$.

Theorem 6.1. *The transformation $f \mapsto \hat{f}$ is a partial isometry from the modified Sobolev space $H^1(a, c)$ onto $B(c)$ with initial subspace $D(c)$.*

Proof. For all $\zeta \in \mathbb{C}$, the transform of the function $f_\zeta = \phi_\zeta^*|_{(a,c)}$ is given by

$$\hat{f}_\zeta(z) = \int_a^c \phi_\zeta(x)^* \phi_z(x) d\chi(x) + \int_a^c \phi_\zeta^{[1]}(x)^* \phi_z^{[1]}(x) d\varsigma(x) = K(\zeta, z, c), \quad z \in \mathbb{C}$$

and hence lies in the de Branges space $B(c)$. Moreover, for some given $\zeta_1, \zeta_2 \in \mathbb{C}$ we have

$$\begin{aligned} \langle f_{\zeta_1}, f_{\zeta_2} \rangle_{H^1(a,c)} &= \int_a^c \phi_{\zeta_1}(x)^* \phi_{\zeta_2}(x) d\chi(x) + \int_a^c \phi_{\zeta_1}^{[1]}(x)^* \phi_{\zeta_2}^{[1]}(x) d\zeta(x) \\ &= K(\zeta_1, \zeta_2, c) = [K(\zeta_1, \cdot, c), K(\zeta_2, \cdot, c)]_{B(c)} = [\hat{f}_{\zeta_1}, \hat{f}_{\zeta_2}]_{B(c)}. \end{aligned}$$

Now, since the functions $K(\zeta, \cdot, c)$, $\zeta \in \mathbb{C}$ are dense in $B(c)$, our transformation uniquely extends to a unitary linear map V from $D(c)$ onto $B(c)$. Moreover, because the functionals $f \mapsto \hat{f}(z)$ and $f \mapsto Vf(z)$ are continuous on $D(c)$ for each fixed $z \in \mathbb{C}$, we conclude that V is nothing but our transform restricted to $D(c)$. Finally, it is easily seen that transforms of functions which are orthogonal to $D(c)$ vanish identically. \square

In the following, the closed linear subspace

$$B^\circ(c) = \{F \in B(c) \mid F(0) = 0\}$$

of functions in $B(c)$ which vanish at zero will be of particular interest. This subspace consists precisely of all transforms of functions in $H^1(a, c)$ which vanish in c . In fact, an integration by parts shows that

$$\hat{f}(0) = \phi_0^{[1]}(c)f(c), \quad f \in H^1(a, c),$$

where the boundary term at a vanishes since ϕ_0 is a scalar multiple of w_a . Moreover, the orthogonal complement of $B^\circ(c)$ consists of all scalar multiples of the function $K(0, \cdot, c)$. Hence it corresponds to the one-dimensional subspace of $D(c)$ spanned by the function $\phi_0|_{(a,c)}$.

The crucial properties of the de Branges spaces $B(c)$, $c \in (a, b)$ only hold if c lies in the support

$$\text{supp}(\varrho) = \overline{\{x \in (a, b) \mid \forall \varepsilon > 0 : |\varrho|((x - \varepsilon, x + \varepsilon)) > 0\}}$$

of ϱ . However, for the proof of our inverse uniqueness result a modified set Σ instead of $\text{supp}(\varrho)$ will be more convenient. This set $\Sigma \subseteq \text{supp}(\varrho) \cup \{a, b\}$ is defined as follows. Take $\text{supp}(\varrho)$ and add a if τ is regular at a , there are no Neumann boundary conditions at a and $|\varrho|$ has no mass near a . Under similar conditions one adds the endpoint b . Moreover, if a has not been added, then remove the point $a_\varrho = \inf \text{supp}(\varrho)$ unless $|\varrho|((a_\varrho, c)) = 0$ for some $c \in (a_\varrho, b)$. Similarly, if b has not been added, then remove the point $b_\varrho = \sup \text{supp}(\varrho)$ unless $|\varrho|((c, b_\varrho)) = 0$ for some $c \in (a, b_\varrho)$. The following lemma gives a hint why this definition might be useful.

Lemma 6.2. *The closure of the domain of S is given by*

$$\mathfrak{D} := \overline{\text{dom}(S)} = \overline{\text{span}\{\delta_c \mid c \in \Sigma\}}.$$

Proof. The multi-valued part of S is given by

$$(*) \quad \text{mul}(S) = \{h \in \text{mul}(T_{\max}) \mid V((0, h), v^*)(a) = V((0, h), w^*)(b) = 0\}.$$

Now if $c \in \Sigma \cap (a, b)$, then $\delta_c \perp \text{mul}(S)$ since each $h \in \text{mul}(T_{\max})$ vanishes almost everywhere with respect to $|\varrho|$. Moreover, if $a \in \Sigma$ then τ is regular at a and there are no Neumann boundary conditions at a . Thus, each $h \in \text{mul}(S)$ vanishes in a in view of $(*)$ and hence $\delta_a \perp \text{mul}(S)$. Similarly one shows that $\delta_b \perp \text{mul}(S)$ provided that $b \in \Sigma$. Hence the closure of the linear span of all functions δ_c , $c \in \Sigma$

is orthogonal to $\text{mul}(S)$ and hence contained in \mathfrak{D} . In order to prove the converse let

$$h \in \overline{\text{span}\{\delta_c \mid c \in \Sigma\}}^\perp.$$

Since h is continuous this implies that h vanishes on $\text{supp}(\varrho)$, hence h lies in $\text{mul}(T_{\max})$. Now suppose that

$$(**) \quad V((0, h), v^*)(a) = \lim_{\alpha \rightarrow a} h(\alpha) v^{[1]}(\alpha)^* \neq 0,$$

then τ is necessarily in the l.c. case at a . If ϱ had mass near a , we would infer that $h(a) = 0$ since h vanishes on $\text{supp}(\varrho)$. Hence τ is even regular at a and $(**)$ implies that there are no Neumann boundary conditions at a . Therefore a lies in Σ and hence $h(a) = \langle h, \delta_a \rangle = 0$, contradicting $(**)$. A similar argument for the right endpoint b shows that h lies in $\text{mul}(S)$, which finishes the proof. \square

Also note that functions in \mathfrak{D} are uniquely determined by their values on Σ . In fact, if $f_1, f_2 \in \mathfrak{D}$ such that $f_1(c) = f_2(c)$, $c \in \Sigma$, then $f_1 - f_2$ lies in the orthogonal complement of \mathfrak{D} in view of Lemma 6.2 and hence $f_1 = f_2$.

Now before we state our main embedding theorem, it remains to introduce the de Branges spaces $B(a)$ if $a \in \Sigma$ and $B(b)$ if $b \in \Sigma$. First of all if $a \in \Sigma$, then let $B(a)$ be the one-dimensional space spanned by the entire function $z \mapsto \phi_z(a)$. It does not matter which inner product this space is equipped with; each one turns $B(a)$ into a de Branges space as is easily seen from [14, Theorem 23]. In particular, note that $B^\circ(a) = \{0\}$. Finally if $b \in \Sigma$, then let $B(b)$ be the de Branges space associated with the de Branges function

$$E(z, b) = z\phi_z(b) + i\phi_z^{[1]}(b), \quad z \in \mathbb{C}.$$

The space $B(b)$ has the same properties as the other de Branges spaces $B(c)$, $c \in (a, b)$. For example the reproducing kernel is given as in (6.2) and Theorem 6.1 holds with c replaced by b .

The following result is basically a consequence of Theorem 6.1 and Proposition 5.3, linking our transformation with the generalized Fourier transform \mathcal{F} . In the following, μ denote the spectral measure associated with the real analytic solutions ϕ_z , $z \in \mathbb{C}^\times$ as constructed in the previous section. However, note that in the present case we may extend μ to a Borel measure on \mathbb{R} by setting $\mu(\{0\}) = 0$.

Theorem 6.3. *For each $c \in \Sigma$ the de Branges space $B(c)$ is a closed subspace of $L^2(\mathbb{R}; \mu)$ with*

$$(6.5) \quad \langle F, G \rangle_\mu = [P^\circ F, P^\circ G]_{B(c)} + \frac{F(0)G(0)^*}{|\phi_0(c)|^2} \|\delta_c\|_{H^1(a,b)}^2, \quad F, G \in B(c),$$

where P° is the orthogonal projection from $B(c)$ onto $B^\circ(c)$.

Proof. First of all note that for $z \in \mathbb{C}$ and $h \in \text{mul}(S) \subseteq \text{mul}(T_{\max})$ we have

$$(*) \quad \int_a^c \phi_z(x) h(x)^* d\chi(x) + \int_a^c \phi_z^{[1]}(x) h^{[1]}(x)^* d\varsigma(x) = \lim_{x \rightarrow a} \phi_z^{[1]}(x) h(x),$$

since h vanishes almost everywhere with respect to $|\varrho|$ (in particular note that $h(c) = 0$). Moreover, the limit on the right-hand side is zero since

$$\lim_{x \rightarrow a} \phi_z^{[1]}(x) h(x) = V((0, h), \phi_z)(a) = 0$$

and both, $(0, h)$ and $(\phi_z, z\phi_z)$ lie in S near a . Now, given some arbitrary functions $f, g \in \text{span}\{\phi_z|_{(a,c)} \mid z \in \mathbb{C}\}$, let

$$f_0(x) = \frac{f(c)}{\phi_0(c)}\phi_0(x) \quad \text{and} \quad f_1(x) = f(x) - f_0(x), \quad x \in (a, c)$$

and similarly for the function g . The extensions \bar{f}_1, \bar{g}_1 of f_1, g_1 , defined by

$$\bar{f}_1(x) = \begin{cases} f_1(x), & \text{if } x \in (a, c], \\ 0, & \text{if } x \in (c, b), \end{cases}$$

and similarly for \bar{g}_1 , lie in $H^1(a, b)$ since $f_1(c) = g_1(c) = 0$. Moreover, these extensions even lie in \mathfrak{D} , because $(*)$ shows that they are orthogonal to $\text{mul}(S)$. Now we get the identity

$$(6.6) \quad \begin{aligned} \langle \hat{f}_1, \hat{g}_1 \rangle_\mu &= \langle \mathcal{F}\bar{f}_1, \mathcal{F}\bar{g}_1 \rangle_\mu = \langle \bar{f}_1, \bar{g}_1 \rangle_{H^1(a,b)} = \langle f_1, g_1 \rangle_{H^1(a,c)} \\ &= [\hat{f}_1, \hat{g}_1]_{B(c)}, \end{aligned}$$

where we used Proposition 5.3, Lemma 5.2 and Theorem 6.1. Moreover, from (6.3) (also note that $\delta_c \in \mathfrak{D}$) we get

$$\langle \hat{f}_0, \hat{g}_0 \rangle_\mu = f_0(c)g_0(c)^* \left| \frac{\phi_0^{[1]}(c)}{\phi_0(c)} \right|^2 \int_{\mathbb{R}} |\phi_\lambda(c)|^2 d\mu(\lambda) = \frac{\hat{f}_0(0)\hat{g}_0(0)^*}{|\phi_0(c)|^2} \|\delta_c\|_{H^1(a,b)}^2.$$

Furthermore,

$$\langle \hat{f}_1, \hat{g}_0 \rangle_\mu = g_0(c)^* \frac{\phi_0^{[1]}(c)}{\phi_0(c)} \int_{\mathbb{R}} \phi_\lambda(c) \hat{f}_1(\lambda) d\mu(\lambda) = g_0(c)^* \frac{\phi_0^{[1]}(c)}{\phi_0(c)} f_1(c) = 0,$$

i.e. the function \hat{g}_0 is orthogonal to \hat{f}_1 not only in $B(c)$ but also in $L^2(\mathbb{R}; \mu)$. Using these properties, we finally obtain

$$\langle \hat{f}, \hat{g} \rangle_\mu = \langle \hat{f}_1, \hat{g}_1 \rangle_\mu + \langle \hat{f}_0, \hat{g}_0 \rangle_\mu = \langle P^\circ \hat{f}, P^\circ \hat{g} \rangle_{B(c)} + \frac{\hat{f}(0)\hat{g}(0)^*}{|\phi_0(c)|^2} \|\delta_c\|_{H^1(a,b)}^2.$$

Hence (6.5) holds for all F, G in a dense subspace of $B(c)$. Now it is quite easy to see that $B(c)$ is actually continuously embedded in $L^2(\mathbb{R}; \mu)$ and that (6.5) holds for all $F, G \in B(c)$. Moreover, $B(c)$ is a closed subspace of $L^2(\mathbb{R}; \mu)$ since the norms $\|\cdot\|_{B(c)}$ and $\|\cdot\|_\mu$ are equivalent on $B(c)$. \square

In particular, note that under the assumption of Theorem 6.3 the subspace $B^\circ(c)$ is isometrically embedded in $L^2(\mathbb{R}; \mu)$. Moreover, the embedding $B(c) \rightarrow L^2(\mathbb{R}; \mu)$ preserves orthogonality and a simple calculation shows that for functions F in the orthogonal complement of $B^\circ(c)$ we have

$$(6.7) \quad \|F\|_{B(c)}^2 = \frac{|F(0)|^2}{\phi_0^{[1]}(c)\phi_0(c)} = \left(1 - \frac{w_b^{[1]}(c)}{w_b(c)} \frac{w_a(c)}{w_a^{[1]}(c)} \right) \|F\|_\mu^2,$$

at least if $c \neq a$. This difference between $B^\circ(c)$ and its orthogonal complement stems from the fact that the functions in $H^1(a, c)$ corresponding to $B^\circ(c)$ are isometrically embedded in $H^1(a, b)$, whereas the functions corresponding to its orthogonal complement are not.

The following results contain further properties of our de Branges spaces which are needed for the inverse uniqueness theorem in the next section. First of all, we will show that they are totally ordered and strictly increasing.

Proposition 6.4. *If $c_1, c_2 \in \Sigma$ with $c_1 < c_2$, then*

$$B(c_1) \subsetneq B(c_2).$$

Moreover, if $|\varrho|((c_1, c_2)) = 0$ then $B(c_1)$ has codimension one in $B(c_2)$.

Proof. If $\delta \in H^1(a, c_2)$ is such that

$$\langle f, \delta \rangle_{H^1(a, c_2)} = f(c_1), \quad f \in H^1(a, c_2),$$

then the modified Sobolev space $H^1(a, c_2)$ may be decomposed into

$$H^1(a, c_2) = H_-^1(a, c_2) \oplus \text{span}\{\delta\} \oplus H_+^1(a, c_2).$$

Here $H_-^1(a, c_2)$ is the subspace of functions in $H^1(a, c_2)$ vanishing on (c_1, c_2) and $H_+^1(a, c_2)$ is the subspace of functions in $H^1(a, c_2)$ vanishing on (a, c_1) . Now the transforms of functions in $H_-^1(a, c_2)$ are precisely the transforms of functions in $H^1(a, c_1)$ which vanish in c_1 , i.e. $B^\circ(c_1)$. The transform of the subspace $\text{span}\{\delta\}$ is precisely the orthogonal complement of $B^\circ(c_1)$. Hence one sees that $B(c_1)$ is contained in $B(c_2)$. In order to prove that $B(c_2)$ is larger indeed, suppose that the function $z \mapsto \phi_z(c_2)$ belongs to $B(c_1)$. Since this function is orthogonal to $B^\circ(c_2)$ it is also orthogonal to $B^\circ(c_1)$ by Theorem 6.3. Thus we infer that the functions $z \mapsto \phi_z(c_1)$ and $z \mapsto \phi_z(c_2)$ are linearly dependent. Now from Lemma 5.2 (hereby also note that δ_{c_1} and δ_{c_2} lie in \mathfrak{D}) one sees that δ_{c_1} and δ_{c_2} are also linearly dependent, which gives a contradiction.

It remains to prove that the space of transforms of functions in $H_+^1(a, c_2)$ is at most one-dimensional provided that $|\varrho|((c_1, c_2)) = 0$. Indeed, for each function $f \in H_+^1(a, c_2)$ an integration by parts shows that

$$\hat{f}(z) = \phi_z^{[1]}(c_2)f(c_2) - \phi_z^{[1]}(c_1)f(c_1) + z \int_{c_1}^{c_2} \phi_z f d\varrho = \phi_z^{[1]}(c_2)f(c_2), \quad z \in \mathbb{C},$$

since f vanishes on $(a, c_1]$ and $|\varrho|((c_1, c_2)) = 0$. \square

The following result shows that our de Branges spaces are continuous in some sense. This is due to the assumption that the measure ς is absolutely continuous with respect to the Lebesgue measure. Otherwise, there would be jumps of dimension one in points where ς has mass.

Proposition 6.5. *If $c, \alpha_n, \beta_n \in \text{supp}(\varrho)$, $n \in \mathbb{N}$ are such that $\alpha_n \uparrow c$ and $\beta_n \downarrow c$ as $n \rightarrow \infty$, then*

$$(6.8) \quad \overline{\bigcup_{n \in \mathbb{N}} B(\alpha_n)} = B(c) = \bigcap_{n \in \mathbb{N}} B(\beta_n),$$

where the closure is taken in $L^2(\mathbb{R}; \mu)$.

Proof. From Proposition 6.4 it is clear that

$$\overline{\bigcup_{n \in \mathbb{N}} B(\alpha_n)} \subseteq B(c) \subseteq \bigcap_{n \in \mathbb{N}} B(\beta_n).$$

If $F \in B^\circ(c)$, then there is an $f \in H^1(a, c)$ with $f(c) = 0$ such that $\hat{f} = F$. Now choose a sequence $f_k \in H^1(a, c)$, $k \in \mathbb{N}$ of functions which vanish near c , such that

$f_k \rightarrow f$ as $k \rightarrow \infty$. By our assumptions the transform of each of these functions lies in $B(\alpha_n)$, provided that $n \in \mathbb{N}$ is large enough, i.e.

$$\hat{f}_k \in \bigcup_{n \in \mathbb{N}} B(\alpha_n), \quad k \in \mathbb{N}.$$

Consequently the transform of f lies in the closure of this union. Moreover, for each $n \in \mathbb{N}$ the entire function $z \mapsto \phi_z(\alpha_n)$ lies in $B(\alpha_n)$. Now since $\delta_{\alpha_n} \rightarrow \delta_c$ in $H^1(a, b)$, Lemma 5.2 shows that the entire function $z \mapsto \phi_z(c)$ lies in the closure of our union which proves the first equality in (6.8).

Next, if $F \in B(\beta_n)$ for each $n \in \mathbb{N}$, then there are $f_n \in D(\beta_n)$ such that

$$F(z) = \int_a^{\beta_n} \phi_z(x) f_n(x) d\chi(x) + \int_a^{\beta_n} \phi_z^{[1]}(x) f_n^{[1]}(x) d\varsigma(x), \quad z \in \mathbb{C}, \quad n \in \mathbb{N}.$$

Moreover, from Theorem 6.1 and Theorem 6.3 we infer

$$\|f_n\|_{H^1(a, \beta_n)}^2 = \|F\|_{B(\beta_n)}^2 \leq \left(1 + \left| \frac{w_b^{[1]}(\beta_n)}{w_b(\beta_n)} \frac{w_a(\beta_n)}{w_a^{[1]}(\beta_n)} \right| \right) \|F\|_\mu^2, \quad n \in \mathbb{N},$$

where the coefficient on the right-hand side is bounded uniformly for all $n \in \mathbb{N}$ by the properties of the solutions w_a and w_b . Hence there is some subsequence of $f_n|_{(a, c)}$, $n \in \mathbb{N}$ converging weakly in $H^1(a, c)$ to say f . Now this yields for all $z \in \mathbb{C}$

$$F(z) = \hat{f}(z) + \chi(\{c\})\phi_z(c)f(c) + \lim_{n \rightarrow \infty} \int_{(c, \beta_n)} \phi_z f_n d\chi + \int_{(c, \beta_n)} \phi_z^{[1]} f_n^{[1]} d\varsigma,$$

where the limit is actually zero. In fact, for each $z \in \mathbb{C}$ and $n \in \mathbb{N}$ we have

$$\left| \int_{(c, \beta_n)} \phi_z f_n d\chi + \int_{(c, \beta_n)} \phi_z^{[1]} f_n^{[1]} d\varsigma \right| \leq C_z^2 \|f_n\|_{H^1(a, \beta_n)} (\chi((c, \beta_n)) + \varsigma((c, \beta_n))),$$

where $C_z \in \mathbb{R}$ is such that the moduli of ϕ_z and $\phi_z^{[1]}$ on (c, β_1) are bounded by C_z . But this shows that F actually is the transform of a function in $H^1(a, b)$ and hence lies in $B(c)$ which finishes the proof. \square

Finally we will prove that our de Branges spaces decrease to zero near a and fill the whole space $L^2(\mathbb{R}; \mu)$ near b .

Proposition 6.6. *The de Branges spaces $B(c)$, $c \in \Sigma$ satisfy*

$$(6.9) \quad \bigcap_{c \in \Sigma} B^\circ(c) = \{0\} \quad \text{and} \quad \overline{\bigcup_{c \in \Sigma} B(c)} = L^2(\mathbb{R}; \mu).$$

Proof. First suppose that $\text{supp}(\varrho) \cap (a, c) \neq \emptyset$ for each $c \in (a, b)$ and pick some $F \in \bigcap_{c \in \Sigma} B^\circ(c)$. Then for each $\zeta \in \mathbb{C}$ we have

$$\begin{aligned} |F(\zeta)| &\leq [F, K(\zeta, \cdot, c)]_{B(c)} \leq \|F\|_{B(c)} [K(\zeta, \cdot, c), K(\zeta, \cdot, c)]_{B(c)} \\ &\leq \|F\|_\mu K(\zeta, \zeta, c) \end{aligned}$$

for each $c \in \text{supp}(\varrho)$. Now from (6.2) we infer that $K(\zeta, \zeta, c) \rightarrow 0$ as $c \rightarrow a$ and hence that $F = 0$. Otherwise, if $\alpha_\varrho = \inf \text{supp}(\varrho) > a$, then the subspace

$$D^\circ(\alpha_\varrho) = \{f \in D(\alpha_\varrho) \mid f(\alpha_\varrho) = 0\},$$

corresponding to $B^\circ(\alpha_\varrho)$, is at most one-dimensional. In fact, this is because each function $\phi_z|_{(a, \alpha_\varrho)}$, $z \in \mathbb{C}$ is a solution of $\tau u = 0$ on (a, α_ϱ) in this case. Consequently, the functions in $D^\circ(\alpha_\varrho)$ are also solutions of $\tau u = 0$ on (a, α_ϱ) . Moreover, if $\varsigma + \chi$

is infinite near a , then each $f \in D^\circ(\alpha_\varrho)$ is a scalar multiple of w_a on (a, α_ϱ) with $f(\alpha_\varrho) = 0$ and hence vanishes identically. Also if $\varsigma + \chi$ is finite near a and there are Neumann boundary conditions at a , one sees that f is a scalar multiple of w_a and hence identically zero. We conclude that the first equality in the claim holds in these cases. Finally, if $\varsigma + \chi$ is finite near a and there are no Neumann boundary conditions at a , then $a \in \Sigma$ and hence clearly $B^\circ(a) = \{0\}$. For the second equality note that the linear span of functions $z \mapsto \phi_z(c)$, $c \in \Sigma$ is dense in $L^2(\mathbb{R}; \mu)$ in view of Lemma 6.2 and Lemma 5.2. \square

Remark 6.7. *At this point let us mention that a real entire solution ϕ_z , $z \in \mathbb{C}$ as in this section is not unique. In fact, any other such solution is given by*

$$\tilde{\phi}_z = e^{g(z)} \phi_z, \quad z \in \mathbb{C}$$

for some real entire function g . The corresponding spectral measures are related by

$$\tilde{\mu}(B) = \int_B e^{-2g(\lambda)} d\mu(\lambda)$$

for each Borel set $B \subseteq \mathbb{R}$. In particular, the measures are mutually absolutely continuous and the associated spectral transforms just differ by a simple rescaling with a positive function. Moreover, from Theorem 6.1 it is easily seen that for each $c \in (a, b)$, multiplication with the entire function e^{-g} maps $B(c)$ isometrically onto the corresponding de Branges space $\tilde{B}(c)$.

7. INVERSE UNIQUENESS RESULTS

The present section is devoted to our inverse uniqueness result. We will prove that the spectral measure determines a left-definite Sturm–Liouville operator up to some Liouville transformation (see e.g. [7] or [3] for the right-definite case). Therefore let S_1 and S_2 be two self-adjoint left-definite Sturm–Liouville relations (with separated boundary conditions), both satisfying the assumptions made in the previous section, i.e. zero is not an eigenvalue of S_1 and S_2 and there are real entire solutions satisfying the boundary condition at the left endpoint. Moreover, again we assume that the measures ς_1 and ς_2 are absolutely continuous with respect to the Lebesgue measure. All remaining quantities corresponding to S_1 respectively S_2 are denoted with an additional subscript.

We will first state a part of the proof of our inverse uniqueness result as a separate lemma. Note that the equality in the claim of this lemma has to be read as sets of entire functions and not as de Branges spaces. In general the norms of these spaces will differ from each other.

Lemma 7.1. *Suppose that the function*

$$(7.1) \quad \frac{E_1(z, x_1)}{E_2(z, x_2)}, \quad z \in \mathbb{C}^+$$

is of bounded type for some $x_1 \in \Sigma_1$ and $x_2 \in \Sigma_2$. If $\mu_1 = \mu_2$, then there is an increasing continuous bijection η from Σ_1 onto Σ_2 such that

$$B_1(x_1) = B_2(\eta(x_1)), \quad x_1 \in \Sigma_1.$$

Proof. First of all note that by the definition of de Branges spaces and Proposition 6.4 the function in (7.1) is of bounded type for all $x_1 \in \Sigma_1$ and $x_2 \in \Sigma_2$. We will first consider the case when Σ_1 consists of finitely many (strictly increasing) points $x_{1,n}$, $n = 1, \dots, N$ separately. In this case $\mu_1 = \mu_2$ is supported on N points, since

\mathcal{F}_1 is a unitary map from \mathfrak{D}_1 onto $L^2(\mathbb{R}; \mu_1)$. Hence, Σ_2 also consists of finitely many (strictly increasing) points $x_{2,n}$, $n = 1, \dots, N$. Now let η be the unique strictly increasing bijection from Σ_1 onto Σ_2 , i.e. $\eta(x_{1,n}) = x_{2,n}$, $n = 1, \dots, N$. Using the properties of our de Branges spaces it is quite simple to see that

$$\dim B_1(x_{1,n}) = \dim B_2(x_{2,n}) = n, \quad n = 1, \dots, N,$$

and therefore the claim follows from Theorem 6.3 and Theorem A.1.

Now suppose that Σ_1 consists of infinitely many points and fix some arbitrary $x_1 \in \Sigma_1 \setminus \{\inf \Sigma_1, \sup \Sigma_1\}$. Then from Theorem 6.3 and Theorem A.1 we infer that for each $x_2 \in \Sigma_2$ either $B_1(x_1) \subseteq B_2(x_2)$ or $B_1(x_1) \supseteq B_2(x_2)$ and hence also $B_1^\circ(x_1) \subseteq B_2^\circ(x_2)$ or $B_1^\circ(x_1) \supseteq B_2^\circ(x_2)$. In order to define $\eta(x_1) \in (a_2, b_2)$ we are first going to show that both of the sets

$$\begin{aligned} J_- &= \{x_2 \in \Sigma_2 \mid B_2(x_2) \subsetneq B_1(x_1)\}, \\ J_+ &= \{x_2 \in \Sigma_2 \mid B_1(x_1) \subsetneq B_2(x_2)\}, \end{aligned}$$

are non-empty. Indeed, if J_- was empty, then $B_1^\circ(x_1) \subseteq B_2^\circ(x_2)$ for each $x_2 \in \Sigma_2$ and hence

$$B_1^\circ(x_1) \subseteq \bigcap_{x_2 \in \Sigma_2} B_2^\circ(x_2) = \{0\},$$

in view of Proposition 6.6. Thus we obtained the contradiction $x_1 = \inf \Sigma_1$, since otherwise there would be some $\tilde{x}_1 \in \Sigma_1$ with $\tilde{x}_1 < x_1$ such that $B_1(\tilde{x}_1) \subsetneq B_1(x_1)$. Furthermore, if J_+ was empty, then $B_2(x_2) \subseteq B_1(x_1)$ for each $x_2 \in \Sigma_2$ and hence

$$L^2(\mathbb{R}; \mu_1) = \overline{\bigcup_{x_2 \in \Sigma_2} B_2(x_2)} \subseteq B_1(x_1) \subseteq L^2(\mathbb{R}; \mu_1).$$

But from this we infer the contradiction $x_1 = \sup \Sigma_1$, since otherwise there would be an $\tilde{x}_1 \in \Sigma_1$ with $\tilde{x}_1 > x_1$ such that $B_1(x_1) \subsetneq B_1(\tilde{x}_1) \subseteq L^2(\mathbb{R}; \mu_1)$. Hence we showed that J_- and J_+ are non-empty. Now, if $J_- = \{a_2\}$ then the space $B_2(\alpha_{\varrho_2})$ is two-dimensional and α_{ϱ_2} does not lie in J_+ since otherwise

$$B_2(a_2) \subsetneq B_1(x_1) \subsetneq B_2(\alpha_{\varrho_2}).$$

Thus in this case we may set $\eta(x_1) = \alpha_{\varrho_2}$ and obtain $B_1(x_1) = B_2(\eta(x_1))$. Furthermore, if $J_+ = \{b_2\}$ then the space $B_2(\beta_{\varrho_2})$ has codimension one in $L^2(\mathbb{R}; \mu)$ and β_{ϱ_2} does not lie in J_- since otherwise

$$B_2(\beta_{\varrho_2}) \subsetneq B_1(x_1) \subsetneq B_2(b_2).$$

Again, we may define $\eta(x_1) = \beta_{\varrho_2}$ and get $B_1(x_1) = B_2(\eta(x_1))$. Now in the remaining cases J_- is bounded from above in (a_2, b_2) with supremum

$$\eta_-(x_1) = \sup J_- \in (a_2, b_2),$$

and J_+ is bounded from below in (a_2, b_2) with infimum

$$\eta_+(x_1) = \inf J_+ \in (a_2, b_2).$$

Moreover, we have $\eta_\pm(x_1) \in \text{supp}(\varrho_2)$ since $J_\pm \setminus \{a_2, b_2\}$ is contained in $\text{supp}(\varrho_2)$. Now Proposition 6.5 shows that

$$B_2(\eta_-(x_1)) \subseteq B_1(x_1) \subseteq B_2(\eta_+(x_1)).$$

If $B_1(x_1) = B_2(\eta_-(x_1))$, set $\eta(x_1) = \eta_-(x_1)$ and if $B_1(x_1) = B_2(\eta_+(x_1))$, set $\eta(x_1) = \eta_+(x_1)$ to obtain $B_1(x_1) = B_2(\eta(x_1))$. Otherwise we have

$$B_2(\eta_-(x_1)) \subsetneq B_1(x_1) \subsetneq B_2(\eta_+(x_1)),$$

and hence $\text{supp}(\varrho_2) \cap (\eta_-(x_1), \eta_+(x_1)) \neq \emptyset$ in view of Proposition 6.4. Now we may choose $\eta(x_1)$ in this intersection and get $B_1(x_1) = B_2(\eta(x_1))$ since $\eta(x_1)$ neither lies in J_- nor in J_+ .

Up to now we constructed a function $\eta : \Sigma_1 \setminus \{\inf \Sigma_1, \sup \Sigma_1\} \rightarrow \Sigma_2$ such that $B_1(x_1) = B_2(\eta(x_1))$ for each $x_1 \in \Sigma_1 \setminus \{\inf \Sigma_1, \sup \Sigma_1\}$. Now if $\inf \Sigma_1$ lies in Σ_1 and we set $x_1 = \inf \Sigma_1 \setminus \{\inf \Sigma_1\}$, then $B_1(x_1) = B_2(\eta(x_1))$ is two-dimensional and from Proposition 6.6 we infer that there is an $x_2 \in \Sigma_2$ with

$$\{0\} \subsetneq B_2(x_2) \subsetneq B_2(\eta(x_1)) = B_1(x_1).$$

Hence we may set $\eta(\inf \Sigma_1) = x_2$ and obtain $B_1(\inf \Sigma_1) = B_2(\eta(\inf \Sigma_1))$. Similarly, if $\sup \Sigma_1$ lies in Σ_1 and we set $x_1 = \sup \Sigma_1 \setminus \{\sup \Sigma_1\}$, then the space $B_1(x_1) = B_2(\eta(x_1))$ has codimension one in $B_1(\sup \Sigma_1) = L^2(\mathbb{R}; \mu_1)$. But because of Proposition 6.6 there is an $x_2 \in \Sigma_2$ such that

$$B_2(\eta(\beta_{e_1})) \subsetneq B_2(x_2) \subseteq L^2(\mathbb{R}; \mu_1).$$

Again, we may define $\eta(\sup \Sigma_1) = x_2$ and get $B_1(\sup \Sigma_1) = B_2(\eta(\sup \Sigma_1))$. Thus, we extended our function η to all of Σ_1 and are left to prove the remaining claimed properties.

The fact that η is increasing is a simple consequence of Proposition 6.4. Now if $x_2 \in \Sigma_2$, then the first part of the proof with the roles of Σ_1 and Σ_2 reversed shows that there is an $x_1 \in \Sigma_1$ with $B_1(x_1) = B_2(x_2) = B_1(\eta(x_1))$. In view of Proposition 6.4 this yields $\eta(x_1) = x_2$ and hence η is a bijection. Finally, continuity follows from Proposition 6.5. Indeed, if $c, c_n \in \Sigma_1, n \in \mathbb{N}$ such that $c_n \uparrow c$ as $n \rightarrow \infty$, then

$$B_2\left(\lim_{n \rightarrow \infty} \eta(c_n)\right) = \overline{\bigcup_{n \in \mathbb{N}} B_2(\eta(c_n))} = \overline{\bigcup_{n \in \mathbb{N}} B_1(c_n)} = B_1(c) = B_2(\eta(c))$$

and hence $\eta(c_n) \rightarrow \eta(c)$ as $n \rightarrow \infty$. Similarly, if $c_n \downarrow c$ as $n \rightarrow \infty$, then

$$B_2\left(\lim_{n \rightarrow \infty} \eta(c_n)\right) = \bigcap_{n \in \mathbb{N}} B_2(\eta(c_n)) = \bigcap_{n \in \mathbb{N}} B_1(c_n) = B_1(c) = B_2(\eta(c))$$

and hence again $\eta(c_n) \rightarrow \eta(c)$ as $n \rightarrow \infty$. \square

Note that the condition that the function in (7.1) is of bounded type is actually equivalent to the function

$$\frac{\phi_{1,z}(x_1)}{\phi_{2,z}(x_2)}, \quad z \in \mathbb{C}^+$$

being of bounded type for some $x_1 \in \Sigma_1$ and $x_2 \in \Sigma_2$. Unfortunately, these conditions are somewhat inconvenient in view of applications. However, note that this assumption is for example fulfilled if for some $x_1 \in \Sigma_1$ and $x_2 \in \Sigma_2$ the entire functions $z \mapsto \phi_{1,z}(x_1)$ and $z \mapsto \phi_{2,z}(x_2)$ are of finite exponential type such that the logarithmic integrals

$$\int_{\mathbb{R}} \frac{\ln^+ |\phi_{j,\lambda}(x_j)|}{1 + \lambda^2} d\lambda < \infty, \quad j = 1, 2$$

are finite. Here \ln^+ is the positive part of the natural logarithm. Indeed, a theorem of Krein [27, Theorem 6.17], [31, Section 16.1] states that in this case the functions $z \mapsto \phi_{j,z}(x_j)$, $j = 1, 2$ (and hence also their quotient) are of bounded type in the upper and in the lower complex half-plane. Moreover, note that the conclusion of Lemma 7.1 is also true if for some (and hence all) $x_1 \in (a_1, b_1)$ and $x_2 \in (a_2, b_2)$ the functions $E_1(\cdot, x_1)$, $E_2(\cdot, x_2)$ are of exponential type zero, i.e.

$$\ln^+ |E_j(z, x_j)| = o(|z|), \quad j = 1, 2$$

as $|z| \rightarrow \infty$ in \mathbb{C} . The proof therefore is literally the same, except that one has to apply Theorem A.2 instead of Theorem A.1.

With all the work done in Lemma 7.1 it is now quite simple to show that the spectral measure determines our self-adjoint Sturm–Liouville relation up to a Liouville transform. Here, a Liouville transform \mathcal{L} is a unitary map from \mathfrak{D}_2 onto \mathfrak{D}_1 given by

$$(7.2) \quad \mathcal{L}f_2(x_1) = \kappa(x_1)f_2(\eta(x_1)), \quad x_1 \in \Sigma_1, \quad f_2 \in \mathfrak{D}_2,$$

where η is an increasing continuous bijection from Σ_1 onto Σ_2 and κ is a non-vanishing real function on Σ_1 . We say that the Liouville transform \mathcal{L} maps S_1 onto S_2 if

$$S_2 = \mathcal{L}^* S_1 \mathcal{L},$$

where \mathcal{L}^* is the adjoint of \mathcal{L} regarded as a linear relation in $H^1(a_2, b_2) \times H^1(a_1, b_1)$. Note that in this case the operator parts of S_1 , S_2 are unitarily equivalent in view of this Liouville transform \mathcal{L} .

Theorem 7.2. *Suppose that the function*

$$\frac{E_1(z, x_1)}{E_2(z, x_2)}, \quad z \in \mathbb{C}^+$$

is of bounded type for some $x_1 \in \Sigma_1$ and $x_2 \in \Sigma_2$. If $\mu_1 = \mu_2$, then there is a Liouville transform \mathcal{L} mapping S_1 onto S_2 .

Proof. By Lemma 7.1 there is an increasing continuous bijection η from Σ_1 onto Σ_2 such that $B_1(x_1) = B_2(\eta(x_1))$ and hence also $B_1^\circ(x_1) = B_2^\circ(\eta(x_1))$ for each $x_1 \in \Sigma_1$. According to Theorem 6.3, for each fixed $x_1 \in \Sigma_1$ the entire functions

$$z \mapsto \phi_{1,z}(x_1) \quad \text{and} \quad z \mapsto \phi_{2,z}(\eta(x_1))$$

are orthogonal to $B_1^\circ(x_1) = B_2^\circ(\eta(x_1))$ in $L^2(\mathbb{R}; \mu_1)$. From this we infer that

$$(7.3) \quad \phi_{1,z}(x_1) = \kappa(x_1)\phi_{2,z}(\eta(x_1)), \quad z \in \mathbb{C}$$

for some $\kappa(x_1) \in \mathbb{R}^\times$ and hence also

$$(*) \quad \mathcal{F}_1 \delta_{1,x_1} = \kappa(x_1) \mathcal{F}_2 \delta_{2,\eta(x_1)}.$$

Now the linear relation

$$\mathcal{L} = \mathcal{F}_1^* \mathcal{F}_2|_{\mathfrak{D}_2}$$

is a unitary mapping from \mathfrak{D}_2 onto \mathfrak{D}_1 by Lemma 5.2 and moreover, equation $(*)$ shows that

$$(\delta_{1,x_1}, \kappa(x_1)\delta_{2,\eta(x_1)}) \in \mathcal{L}^* = \mathcal{F}_2^{-1} \mathcal{F}_1, \quad x_1 \in \Sigma_1.$$

From this one sees that the transform of some function $f_2 \in \mathfrak{D}_2$ is given by

$$\begin{aligned}\mathcal{L}f_2(x_1) &= \langle \mathcal{L}f_2, \delta_{1,x_1} \rangle_{H^1(a_1,b_1)} = \kappa(x_1) \langle f_2, \delta_{2,\eta(x_1)} \rangle_{H^1(a_2,b_2)} \\ &= \kappa(x_1) f_2(\eta(x_1))\end{aligned}$$

at each point $x_1 \in \Sigma_1$. Finally, we conclude that

$$\begin{aligned}S_2 &= \mathcal{F}_2^{-1} \mathbf{M}_{\text{id}} \mathcal{F}_2|_{\mathfrak{D}_2} = \mathcal{F}_2^{-1} \mathcal{F}_1 \mathcal{F}_1^{-1} \mathbf{M}_{\text{id}} \mathcal{F}_1 \mathcal{F}_1^* \mathcal{F}_2|_{\mathfrak{D}_2} = \mathcal{L}^* \mathcal{F}_1^{-1} \mathbf{M}_{\text{id}} \mathcal{F}_1 \mathcal{L} \\ &= \mathcal{L}^* S_1 \mathcal{L},\end{aligned}$$

from Theorem 5.4. \square

We will now show to which extend the spectral measure determines the coefficients. For the proof we need a result on the high energy asymptotics of solutions of our differential equation (see e.g. [4, Section 6]). Henceforth we will denote with r_j , $j = 1, 2$ the densities of the absolute continuous parts of ϱ_j with respect to the Lebesgue measure and with p_j^{-1} , $j = 1, 2$ the densities of ς_j with respect to the Lebesgue measure.

Lemma 7.3. *For each $j = 1, 2$ and all points $x_j, \tilde{x}_j \in (a_j, b_j)$ we have the asymptotics*

$$\sqrt{\frac{2}{y}} \ln \frac{|\phi_{j,iy}(x_j)|}{|\phi_{j,iy}(\tilde{x}_j)|} \rightarrow \int_{\tilde{x}_j}^{x_j} \sqrt{\frac{|r_j(x)|}{p_j(x)}} dx,$$

as $y \rightarrow \infty$ in \mathbb{R}^+ .

Proof. By our assumptions, the Lebesgue decomposition of the measure ϱ_j with respect to ς_j is given by

$$\varrho_j = r_j p_j \varsigma_j + \varrho_{j,s},$$

where $\varrho_{j,s}$ is the singular part of ϱ_j with respect to the Lebesgue measure. Now the results in [4, Section 6] show that (the square root is the principal one with branch cut along the negative real axis)

$$\begin{aligned}\ln \frac{|\phi_{j,iy}(x_j)|}{|\phi_{j,iy}(\tilde{x}_j)|} &= \text{Re} \left(\int_{\tilde{x}_j}^{x_j} \sqrt{-iy r_j(x) p_j(x)} d\varsigma_j(x) + o(\sqrt{y}) \right) \\ &= \sqrt{\frac{y}{2}} \int_{\tilde{x}_j}^{x_j} \sqrt{\frac{|r_j(x)|}{p_j(x)}} dx + o(\sqrt{y}),\end{aligned}$$

as $y \rightarrow \infty$ in \mathbb{R}^+ , which yields the claim. \square

We are now able to establish a relation between the measure coefficients. However, this is only possible on sets where the support of the weight measure has enough density. Otherwise there would be too much freedom for the remaining coefficients.

Corollary 7.4. *Let $\alpha_1, \beta_1 \in (a_1, b_1)$ with $\alpha_1 < \beta_1$ such that $r_1 \neq 0$ almost everywhere on (α_1, β_1) and $r_2 \neq 0$ almost everywhere on $(\eta(\alpha_1), \eta(\beta_1))$ with respect to the Lebesgue measure. If the function*

$$\frac{E_1(z, x_1)}{E_2(z, x_2)}, \quad z \in \mathbb{C}^+$$

is of bounded type for some $x_1 \in (a_1, b_1)$, $x_2 \in (a_2, b_2)$ and $\mu_1 = \mu_2$, then the functions η and κ from the Liouville transform of Theorem 7.2 satisfy

$$\eta' = \sqrt{\frac{p_2 \circ \eta}{p_1} \frac{|r_1|}{|r_2 \circ \eta|}} \quad \text{and} \quad \kappa^2 = \sqrt{\frac{p_2 \circ \eta}{p_1} \frac{|r_2 \circ \eta|}{|r_1|}}$$

almost everywhere on (α_1, β_1) with respect to the Lebesgue measure and for the measure coefficients we have

$$\varsigma_2 \circ \eta = \kappa^{-2} \varsigma_1, \quad \varrho_2 \circ \eta = \kappa^2 \varrho_1 \quad \text{and} \quad \chi_2 \circ \eta = \kappa^2 \chi_1 - \kappa \kappa^{[1]'},$$

as measures on (α_1, β_1) .

Proof. From equation (7.3) and the asymptotics in Lemma 7.3 we infer that

$$\int_{\tilde{x}_1}^{x_1} \sqrt{\frac{|r_1(x)|}{p_1(x)}} dx = \int_{\eta(\tilde{x}_1)}^{\eta(x_1)} \sqrt{\frac{|r_2(x)|}{p_2(x)}} dx, \quad x_1, \tilde{x}_1 \in (\alpha_1, \beta_1).$$

In view of the Banach–Zareckiĭ theorem (see e.g. [28, Chapter IX; Theorem 4], [22, Theorem 18.25]) this shows that η is locally absolutely continuous on (α_1, β_1) with derivative given as in the claim. More precisely, this follows from an application of [28, Chapter IX; Exercise 13] and [28, Chapter IX; Theorem 5]. Furthermore, since $\phi_{1,0}$, $\phi_{2,0}$ are scalar multiples of $w_{1,a}$, $w_{2,a}$ respectively, we also have

$$(*) \quad w_{1,a}(x_1) = C_a \kappa(x_1) w_{2,a}(\eta(x_1)), \quad x_1 \in (\alpha_1, \beta_1)$$

for some constant $C_a \in \mathbb{R}^\times$. In particular, this shows that κ is locally absolutely continuous on (α_1, β_1) . In fact, the substitution rule for Lebesgue–Stieltjes integrals (see e.g. [19]) shows that

$$\begin{aligned} w_{2,b}(\eta(x_1)) - w_{2,b}(\eta(\tilde{x}_1)) &= \int_{\eta(\tilde{x}_1)}^{\eta(x_1)} w_{2,b}^{[1]} d\varsigma_2 \\ &= \int_{\tilde{x}_1}^{x_1} w_{2,b}^{[1]} \circ \eta d\varsigma_2 \circ \eta, \quad x_1, \tilde{x}_1 \in (\alpha_1, \beta_1) \end{aligned}$$

and hence the function $x_1 \mapsto w_{2,b}(\eta(x_1))$ is locally of bounded variation on (α_1, β_1) . Therefore, from [28, Chapter IX; Theorem 5] we infer that this function is even locally absolutely continuous on (α_1, β_1) and hence so is κ . Moreover, in view of Lemma 5.2, equation (7.3) yields

$$\kappa(x_1)^2 = \frac{\|\delta_{1,x_1}\|_{H^1(a_1,b_1)}^2}{\|\delta_{2,\eta(x_1)}\|_{H^1(a_2,b_2)}^2} = \frac{W(w_{2,b}, w_{2,a})}{W(w_{1,b}, w_{1,a})} \frac{w_{1,a}(x_1) w_{1,b}(x_1)}{w_{2,a}(\eta(x_1)) w_{2,b}(\eta(x_1))}$$

for each $x_1 \in (\alpha_1, \beta_1)$. Inserting $(*)$ we get from this equation

$$w_{1,b}(x_1) = C_a^{-1} \frac{W(w_{1,b}, w_{1,a})}{W(w_{2,b}, w_{2,a})} \kappa(x_1) w_{2,b}(\eta(x_1)), \quad x_1 \in (\alpha_1, \beta_1).$$

Plugging this expression and equation $(*)$ into the definition of the Wronskian $W(w_{1,b}, w_{1,a})$ one obtains

$$1 = \frac{\kappa(x_1)^2 \eta'(x_1) p_1(x_1)}{p_2(\eta(x_1))}, \quad x_1 \in (\alpha_1, \beta_1),$$

which shows that κ is given as in the claim. Next, differentiating equation (7.3) yields

$$\kappa(x_1)\phi_{1,z}^{[1]}(x_1) = \kappa^{[1]}(x_1)\phi_{1,z}(x_1) + \phi_{2,z}^{[1]}(\eta(x_1)), \quad x_1 \in (\alpha_1, \beta_1)$$

for each $z \in \mathbb{C}$. From this we get for all $\alpha, \beta \in (\alpha_1, \beta_1)$

$$\begin{aligned} \int_{\alpha}^{\beta} \phi_{1,z} \kappa d\chi_1 - z \int_{\alpha}^{\beta} \phi_{1,z} \kappa d\varrho_1 &= \\ &= \int_{\alpha}^{\beta} \phi_{1,z} d\kappa^{[1]} + \int_{\alpha}^{\beta} \phi_{1,z} \kappa^{-1} d\chi_2 \circ \eta - z \int_{\alpha}^{\beta} \phi_{1,z} \kappa^{-1} d\varrho_2 \circ \eta, \end{aligned}$$

where we used the integration by parts formula (1.3), the differential equation and the substitution rule. In particular, choosing $z = 0$ this shows that the coefficients χ_1 and χ_2 are related as in the claim (note that $\phi_{1,0}$ does not have any zeros). Using this relation, one sees from the previous equation that for each $z \in \mathbb{C}^{\times}$ and $\alpha, \beta \in (\alpha_1, \beta_1)$ we actually have

$$\int_{\alpha}^{\beta} \phi_{1,z} \kappa d\varrho_1 = \int_{\alpha}^{\beta} \phi_{1,z} \kappa^{-1} d\varrho_2 \circ \eta.$$

Now since for each $x_1 \in (\alpha_1, \beta_1)$ there is some $z \in \mathbb{C}^{\times}$ such that $\phi_{1,z}(x_1) \neq 0$, this shows that the coefficients ϱ_1 and ϱ_2 are related as in the claim. \square

In particular, note that these relations among our measures show that under the assumptions of Corollary 7.4, for every $z \in \mathbb{C}$ and each solution u_2 of $(\tau_2 - z)u = 0$, the function

$$u_1(x_1) = \kappa(x_1)u_2(\eta(x_1)), \quad x_1 \in (\alpha_1, \beta_1)$$

is a solution of $(\tau_1 - z)u = 0$ on (α_1, β_1) . Moreover, linear independence is preserved under this transformation.

In the remaining part of this section we will prove one more inverse uniqueness result, tailor-made to fit the requirements of the isospectral problem of the Camassa–Holm equation. There, we do not want the measures ϱ_1 and ϱ_2 to necessarily have dense support; hence we can not apply Corollary 7.4. However, we will assume that the intervals and the coefficients on the left-hand side of the differential equation are fixed, i.e.

$$a := a_1 = a_2, \quad b := b_1 = b_2, \quad \varsigma := \varsigma_1 = \varsigma_2 \quad \text{and} \quad \chi := \chi_1 = \chi_2,$$

and that τ_1 and τ_2 are in the l.p. case at both endpoints. Another crucial additional assumption we will make for this inverse uniqueness result is that the norms of point evaluations (note that the modified Sobolev spaces are the same for both relations) $\|\delta_c\|_{H^1(a,b)}$ are independent of $c \in (a, b)$. For example this is the case when ς and χ are scalar multiples of the Lebesgue measure, as it is the case for the isospectral problem of the Camassa–Holm equation. Moreover, we suppose that our real entire solutions $\phi_{1,z}$ and $\phi_{2,z}$ coincide at $z = 0$, i.e.

$$(7.4) \quad \phi_{1,0}(x) = \phi_{2,0}(x), \quad x \in (a, b).$$

As a consequence of these assumptions, the coefficient of the second term on the right-hand side of (6.5) in Theorem 6.3 is the same for both problems. Now the weight measure on the right-hand side of our differential equation is uniquely determined by the spectral measure. In view of application to the isospectral problem of the Camassa–Holm equation we state this result with the assumption that our

de Branges functions are of exponential type zero. Of course the same result holds if their quotient is of bounded type in the upper complex half-plane.

Theorem 7.5. *Suppose that $E_1(\cdot, c)$ and $E_2(\cdot, c)$ are of exponential type zero for some $c \in (a, b)$. If $\mu_1 = \mu_2$, then we have $\varrho_1 = \varrho_2$ and $S_1 = S_2$.*

Proof. The (remark after the) proof of Lemma 7.1 shows that there is an increasing continuous bijection η from Σ_1 onto Σ_2 such that

$$B_1(x_1) = B_2(\eta(x_1)), \quad x_1 \in \Sigma_1.$$

Moreover, the proof of Theorem 7.2 (see equation (7.3)) shows that

$$\phi_{1,z}(x_1) = \kappa(x_1)\phi_{2,z}(\eta(x_1)), \quad z \in \mathbb{C}, \quad x_1 \in \Sigma_1$$

for some non-zero real function κ on Σ_1 . In particular, from Lemma 5.2 we infer for each $x_1 \in \Sigma_1$

$$\|\delta_{x_1}\|_{H^1(a,b)}^2 = \|\mathcal{F}_1\delta_{x_1}\|_{\mu_1}^2 = \|\kappa(x_1)\mathcal{F}_2\delta_{\eta(x_1)}\|_{\mu_1}^2 = \kappa(x_1)^2\|\delta_{\eta(x_1)}\|_{H^1(a,b)}^2$$

and hence $\kappa(x_1)^2 = 1$ in view of our additional assumptions. Moreover, Theorem 6.3 shows that $B_1(x_1)$ and $B_2(\eta(x_1))$ actually have the same norm and hence

$$\phi_{1,0}^{[1]}(x_1)\phi_{1,0}(x_1) = K_1(0, 0, x_1) = K_2(0, 0, \eta(x_1)) = \phi_{1,0}^{[1]}(\eta(x_1))\phi_{1,0}(\eta(x_1)).$$

Now since the function $\phi_{1,0}^{[1]}\phi_{1,0}$ is strictly increasing on (a, b) we infer that $\eta(x_1) = x_1$, $x_1 \in \Sigma_1$ and in particular $\Sigma_1 = \Sigma_2$. Hence we even have (note that (7.4) prohibits $\kappa(x_1) = -1$ for some $x_1 \in \Sigma_1$)

$$(*) \quad \phi_{1,z}(x_1) = \phi_{2,z}(x_1), \quad x_1 \in \Sigma_1, \quad z \in \mathbb{C}.$$

Moreover, if (α, β) is a gap of Σ_1 , i.e. $\alpha, \beta \in \Sigma_1$ but $(\alpha, \beta) \cap \Sigma_1 = \emptyset$, then both of this functions are solutions to the same differential equation which coincide on the boundary of the gap. Since their difference is a solution of $\tau_1 u = 0$ which vanishes on the boundary of the gap, we infer that $(*)$ holds for all x_1 in the convex hull of Σ_1 in view of monotonicity of the functions in (2.7). Now if $\underline{x} = \inf \Sigma_1 > a$, then $\varsigma + \chi$ is infinite near a and for each $z \in \mathbb{C}$ the solutions $\phi_{1,z}$ and $\phi_{2,z}$ are scalar multiples of w_a on (a, \underline{x}) . Since they are equal in the point \underline{x} we infer that $(*)$ also holds for all x_1 below \underline{x} . Similarly, if $\bar{x} = \sup \Sigma_1 < b$, then the spectrum of S_1 (and hence also of S_2) is purely discrete. Indeed, the solutions $\psi_{1,b,z}$, $z \in \mathbb{C}$ of $(\tau_1 - z)u = 0$ which are equal to w_b near b are real entire and lie in S_1 near b . Now for each eigenvalue $\lambda \in \mathbb{R}^\times$ the solutions $\phi_{1,\lambda}$ and $\phi_{2,\lambda}$ are scalar multiples of w_b on (\bar{x}, b) . As before we infer that $(*)$ holds for $z = \lambda$ and all $x_1 \in (a, b)$. Finally, from the differential equation we get for each $\alpha, \beta \in (a, b)$ with $\alpha < \beta$

$$\begin{aligned} \lambda \int_\alpha^\beta \phi_{1,\lambda} d\varrho_1 &= -\phi_{1,\lambda}^{[1]}(\beta) + \phi_{1,\lambda}^{[1]}(\alpha) + \int_\alpha^\beta \phi_{1,\lambda} d\chi = \lambda \int_\alpha^\beta \phi_{2,\lambda} d\varrho_2 \\ &= \lambda \int_\alpha^\beta \phi_{1,\lambda} d\varrho_2 \end{aligned}$$

for each $\lambda \in \sigma(S_1)$. But this shows $\varrho_1 = \varrho_2$ and hence also $S_1 = S_2$. Hereby note that for each $x \in (a, b)$ there is an eigenvalue $\lambda \in \mathbb{R}$ such that $\phi_{1,\lambda}(x) \neq 0$. Indeed, otherwise we had $f(x) = 0$ for each $f \in \mathfrak{D}_1$, which is not possible unless $\Sigma_1 = \emptyset$. \square

Note that the condition that the differential expressions are in the l.p. case may be relaxed. For example it is sufficient to assume that τ_j , $j = 1, 2$ are in the l.p. case at a unless $\inf \Sigma_j = a$ and in the l.p. case at b unless $\sup \Sigma_j = b$. The proof therefore is essentially the same.

APPENDIX A. HILBERT SPACES OF ENTIRE FUNCTIONS

In this appendix we will briefly summarize some results of de Branges' theory of Hilbert spaces of entire functions as far as it is needed for the proof of our inverse uniqueness theorem. For a detailed discussion we refer to de Branges' book [14]. First of all recall that an analytic function N in the upper open complex half-plane \mathbb{C}^+ is said to be of bounded type if it can be written as the quotient of two bounded analytic functions. For such a function the number

$$\limsup_{y \rightarrow \infty} \frac{\ln |N(iy)|}{y} \in [-\infty, \infty)$$

is referred to as the mean type of N .

A de Branges function is an entire function E , which satisfies the estimate

$$|E(z)| > |E(z^*)|, \quad z \in \mathbb{C}^+.$$

Associated with such a function is a de Branges space B . It consists of all entire functions F such that

$$\int_{\mathbb{R}} \frac{|F(\lambda)|^2}{|E(\lambda)|^2} d\lambda < \infty$$

and such that F/E and $F^\# / E$ are of bounded type in \mathbb{C}^+ with non-positive mean type. Here $F^\#$ is the entire function given by

$$F^\#(z) = F(z^*)^*, \quad z \in \mathbb{C}.$$

Equipped with the inner product

$$[F, G] = \frac{1}{\pi} \int_{\mathbb{R}} \frac{F(\lambda)G(\lambda)^*}{|E(\lambda)|^2} d\lambda, \quad F, G \in B,$$

the vector space B turns into a reproducing kernel Hilbert space (see [14, Theorem 21]). For each $\zeta \in \mathbb{C}$, the point evaluation in ζ can be written as

$$F(\zeta) = [F, K(\zeta, \cdot)], \quad F \in B,$$

where the reproducing kernel K is given by (see [14, Theorem 19])

$$(A.1) \quad K(\zeta, z) = \frac{E(z)E^\#(\zeta^*) - E(\zeta^*)E^\#(z)}{2i(\zeta^* - z)}, \quad \zeta, z \in \mathbb{C}.$$

Note that though there is a multitude of de Branges functions giving rise to the same de Branges space (including norms), the reproducing kernel K is independent of the actual de Branges function.

One of the main results in de Branges' theory is the subspace ordering theorem; [14, Theorem 35]. For our application we need to slightly weaken the assumptions of this theorem. In order to state it let E_1, E_2 be two de Branges functions with no real zeros and B_1, B_2 be the associated de Branges spaces.

Theorem A.1. *Suppose B_1, B_2 are homeomorphically embedded in $L^2(\mathbb{R}; \mu)$ for some Borel measure μ on \mathbb{R} . If E_1/E_2 is of bounded type in the open upper complex half-plane, then B_1 contains B_2 or B_2 contains B_1 .*

Proof. If a de Branges space B is homeomorphically embedded in $L^2(\mathbb{R}; \mu)$, then B equipped with the inner product inherited from $L^2(\mathbb{R}; \mu)$ is a de Branges space itself. In fact, this is easily verified using the characterization of de Branges spaces in [14, Theorem 23]. Hence, without loss of generality we may assume that B_1, B_2 are isometrically embedded in $L^2(\mathbb{R}; \mu)$ and thus apply [14, Theorem 35]. Therefore, also note that F_1/F_2 is of bounded type in the upper complex half-plane for all $F_1 \in B_1, F_2 \in B_2$ and hence so is the quotient of any corresponding de Branges functions. \square

Note that the isometric embedding in [14, Theorem 35] is only needed to deduce that the smaller space is actually a de Branges subspace of the larger one. The inclusion part is valid under much more general assumptions; see [29, Theorem 5] or [30, Theorem 3.5].

Adapting the proof of [14, Theorem 35], one gets a version of de Branges' ordering theorem, where the bounded type condition is replaced by the assumption that the functions E_1, E_2 are of exponential type zero. Actually this has been done in [26] with the spaces B_1, B_2 being isometrically embedded in some $L^2(\mathbb{R}; \mu)$. Again this latter assumption can be weakened.

Theorem A.2. *Suppose B_1, B_2 are homeomorphically embedded in $L^2(\mathbb{R}; \mu)$ for some Borel measure μ on \mathbb{R} . If E_1, E_2 are of exponential type zero, then B_1 contains B_2 or B_2 contains B_1 .*

Proof. As in the proof of Theorem A.1, the claim can be reduced to the case where the de Branges spaces are isometrically embedded in $L^2(\mathbb{R}; \mu)$. Therefore, also note that a de Branges function is of exponential type zero if and only if all functions in the corresponding de Branges space are (see e.g. [24, Theorem 3.4]). \square

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